Tangents of a parametric curve. We have learned how to write a curve parametrically, as the path of a particle whose position at time t is given by two coordinate functions $(x(t), y(t))$ over a time interval $t \in [a, b]$.

Considering the curve as a track on which the particle runs, the *tangent line* at a point $(x(c), y(c))$ is the path the particle would take if it were suddenly released from the track at time $t = c$, keeping a constant velocity from that moment. The velocity at $t = c$ has horizontal and vertical components $(x'(c), y'(c))$, giving the parametric line:

$$(x(c) + x'(c)\,t,\ y(c) + y'(c)\,t)\,.$$

This is the line which best approximates the curve near the point of tangency $(x(c), y(c))$, with the time set so that the line passes through the point at $t = 0$.

We can convert this parametric line into an xy-equation as in §10.1. The slope is the horizontal over the vertical velocity: $m = \frac{y'(c)}{x'(c)}$, and we know the line passes through $(x(c), y(c))$, so we have the point-slope equation:

$$y = \tfrac{y'(c)}{x'(c)}(x - x(c)) + y(c)\,.$$

Here (x, y) is a general point of the line, but $x(c), y(c), x'(c), y'(c)$ are constants computed from the coordinate functions of the original curve.

To further explain this, we imagine the original curve as the graph of a function $y = f(x)$, meaning $y(t) = f(x(t))$ for all t. The Chain Rule gives:

$$y'(t) \;=\; f'(x(t)) \cdot x'(t) \qquad \Longleftrightarrow \qquad \frac{dy}{dt} \;=\; \frac{dy}{dx} \cdot \frac{dx}{dt}\,.$$

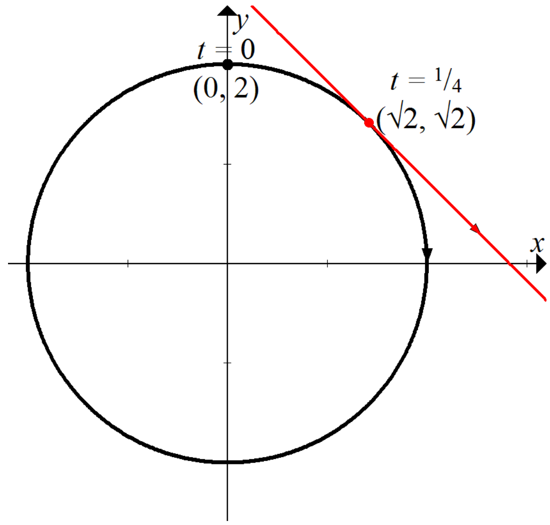
At time $t = c$ and $x = x(c)$, this gives our previous slope formula:

$$f'(x(c)) = \frac{y'(c)}{x'(c)} \qquad \Longleftrightarrow \qquad \frac{dy}{dx} \;=\; \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\,.$$

Tangents of a circle. We find the tangent line to $(x(t), y(t)) = (2\sin(\pi t), 2\cos(\pi t))$ at the point $(\sqrt{2}, \sqrt{2})$. First, to picture the curve, we note:

- Since the components are $2\sin$ and $2\cos$ of the same quantity, the curve is a circle of radius 2.
- The full circle is traced by $\pi t \in [0, 2\pi]$, i.e. $t \in [0, 2]$.
- The curve starts at $(x(0), y(0)) = (0, 2)$ on the y-axis; it moves clockwise, since the x-coordinate $2\sin(\pi t)$ increases for small $t \geq 0$.

Notes by Peter Magyar `magyar@math.msu.edu`



To apply our formulas, we need to know the value $t = c$ at which the curve passes through the given point: $(x(t), y(t)) = (\sqrt{2}, \sqrt{2})$. That is, we must solve the system of equations:

$$\begin{cases} 2\sin(\pi t) = \sqrt{2} \\ 2\cos(\pi t) = \sqrt{2} \end{cases} \qquad \Longleftrightarrow \qquad t = \tfrac{1}{4}\,.$$

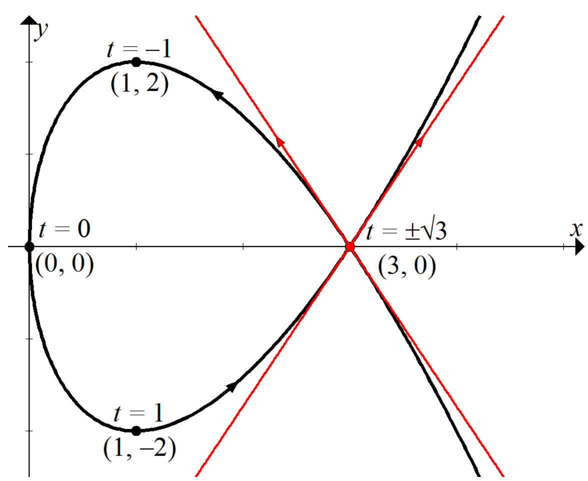
We can find a simultaneous solution to both equations precisely because the point lies on the curve. We have $(x'(c), y'(c)) = (2\pi\cos(\frac{\pi}{4}), -2\pi\sin(\frac{\pi}{4})) = (\sqrt{2}\pi, -\sqrt{2}\pi)$, so the tangent line is:

$$(x(c) + x'(c)t, y(c) + y'(c)t) \;=\; (\sqrt{2} + \sqrt{2}\pi t, \sqrt{2} - \sqrt{2}\pi t)$$

$$y \;=\; \tfrac{y'(c)}{x'(c)}(x - x(c)) + y(c) \;=\; \tfrac{\sqrt{2}\pi}{-\sqrt{2}\pi}(x - \sqrt{2}) + \sqrt{2} \qquad \Longleftrightarrow \qquad y = -x + 2\sqrt{2}\,.$$

Note that each tangent to the circle is perpendicular to the corresponding radius.

Tangents of a polynomial curve. Find the tangent to $(x(t), y(t)) = (t^2, t^3 - 3t)$ at the point $(3, 0)$. This is not a familiar curve, so to picture it, we must plot points by plugging in various values of t:



We see that the curve passes twice through the given point $(3,0)$. Algebraically:

$$\begin{cases} t^2 = 3 \\ t^3 - 3t = 0 \end{cases} \qquad \Longleftrightarrow \qquad t = \sqrt{3} \ \text{ or } \ t = -\sqrt{3}\,.$$

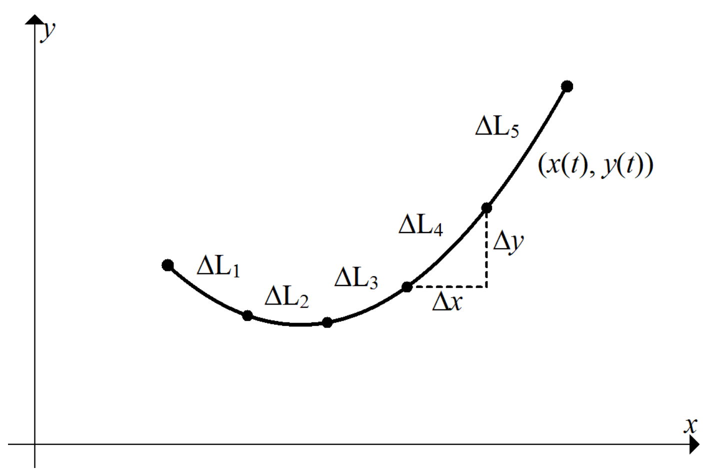
Note that $t^3 - 3t = 0$ by itself has the solutions $t = 0, \pm\sqrt{3}$, but $t = 0$ does not satisfy the first equation $t^2 = 3$: for time $t = 0$, the curve is at $(0,0)$, not $(3,0)$.

Now we can easily find the two tangent lines: $(3 + 2\sqrt{3}t, 6t)$ and $(3 - 2\sqrt{3}t, 6t)$.

EXAMPLE: Which points of this curve have horizontal tangents? The tangent is horizontal when the vertical velocity is zero: $(t^3 - 3t)' = 3t^2 - 3 = 0 \iff t = \pm 1$, corresponding to the points $(1,-2)$ and $(1,2)$.

Arclength. After applying derivatives to parametric curves, we now apply integrals, which compute the size or bulk of geometric objects. The most natural measure of the size of a curve is its arclength. We already computed this for graph curves $y = f(x)$ in §8.1, and now we do the more general parametric case.

We follow the general scheme for computing any measure of size of a geometric object from §5.2. We want the arclength L of a parametric curve $(x(t), y(t))$ for $t \in [a,b]$. We cut the curve into n bits determined by Δt-increments of $t \in [a,b]$.



Because the bit at the sample point t_i is so short, it is well approximated by a straight segment, and we can use the Pythagorean Theorem to compute its length:

$$\Delta L_i \ \approx \ \sqrt{(\Delta x)^2 + (\Delta y)^2} \ = \ \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}\ \Delta t \ = \ \sqrt{(\tfrac{\Delta x}{\Delta t})^2 + (\tfrac{\Delta y}{\Delta t})^2}\ \Delta t\,.$$

In the limit as $n \to \infty$, we get $\Delta t \to 0$ and $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt} = x'(t_i)$; similarly for $\frac{\Delta y}{\Delta t}$:

$$L \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} \Delta L_i \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\tfrac{\Delta x}{\Delta t})^2 + (\tfrac{\Delta y}{\Delta t})^2}\ \Delta t \ = \ \int_a^b \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\ dt\,.$$

In Newton notation:

$$L \ = \ \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt\,.$$

In fact, the integrand is just the total speed of the particle at time t, combining the horizontal and vertical speeds. To be precise, L is the *total distance traveled* along the curve; if the trajectory repeats its motion around a closed loop, or if it changes direction and backtracks, this is larger than the geometric length of the curve.

EXAMPLE: Compute the circumference length of a circle of radius r. The standard parametrization is $(x(t), y(t)) = (r\cos(t), r\sin(t))$ for $t \in [0, 2\pi]$, with derivative $(x'(t), y'(t)) = (-r\sin(t), r\cos(t))$, and length:

$$L = \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt = \int_0^{2\pi} r\sqrt{\sin^2(t) + \cos^2(t)}\, dt$$

$$= \int_0^{2\pi} r\, dt = rt\Big|_{t=0}^{t=2\pi} = 2\pi r\,.$$

The integral is so easy because the particle travels at constant speed r. This was much harder in §8.1, using our previous formula $L = \int_{-r}^{r} \sqrt{1+f'(x)^2}\; dx$, where $f(x) = \sqrt{r^2 - x^2}$.

EXAMPLE: Find the length of one arch of the cycloid from §10.1: $(x(t), y(t)) = (t - \sin(t), 1 - \cos(t))$ for $t \in [0, 2\pi]$. We have $(x'(t), y'(t)) = (1 - \cos(t),\ \sin(t))$, so:

$$L = \int_0^{2\pi} \sqrt{(1 - \cos(t))^2 + (\sin(t))^2}\, dt = \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{2(1 - \cos(t))}\, dt = \int_0^{2\pi} 2\sin(\tfrac{t}{2})\, dt = 8\,.$$

Here we used the identity $\sin(\frac{t}{2}) = \sqrt{\frac{1-\cos(t)}{2}}$.

Area. Consider a parametric curve $(x(t), y(t))$ for $t \in [a, b]$ which closes into a loop starting and ending at the point $(x(a), y(a)) = (x(b), y(b))$. Then the enclosed area is:

$$A = \pm \int_a^b y(t)x'(t)\, dt = \mp \int_a^b x(t)y'(t)\, dt,$$

where the signs after the two equalities are $+, -$ if the loop travels clockwise around the region, but $-, +$ if it travels counterclockwise.

As an exercise, prove these formulas by slice analysis. The point is that area is added while traveling in one direction, then subtracted while traveling back.