# Math 132 Precalculus Review

## Algebra

1. Whole numbers. Division with remainder: $a \div b = q$ rem $r$ means $\frac{a}{b} = q + \frac{r}{b}$

2. Signs: $-(-a) = a, \quad (-a)b = -ab, \quad (-a)(-b) = ab$
$$a - b = a + (-b), \qquad -(a-b) = b - a$$

3. Fractions:
$$\frac{a}{b} = a \cdot \frac{1}{b} \qquad \frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{1}{c/d} = \frac{d}{c} \qquad \frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

4. Distributive law: $a(b+c) = ab + ac$
Hence: $\frac{b+c}{a} = \frac{1}{a}(b+c) = \frac{b}{a} + \frac{c}{a}, \quad -(b+c) = (-1)(b+c) = -b - c$
Mistakes: $\sqrt{x^2+9} \neq x + 3, \quad (x+3)^2 \neq x^2 + 9, \quad \frac{1}{x+3} \neq \frac{1}{x} + \frac{1}{3}$

5. Powers: $a^n = a \cdots a$ ($n$ factors), $a^n a^m = a^{n+m}$, $(ab)^n = a^n b^n$, $(a^n)^m = a^{nm}$
$$a^0 = 1, \quad a^{-n} = \tfrac{1}{a^n}, \quad a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = \sqrt[n]{a^m}, \quad a^{-m/n} = \tfrac{1}{\sqrt[n]{a^m}}$$

6. Difference of powers: $a^2 - b^2 = (a-b)(a+b), \quad a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$$
Sum of geometric series: $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$ (take $a = 1$, $b = r$)

7. Binomial powers: $(a \pm b)^2 = a^2 \pm 2ab + b^2, \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
Binomial Theorem:
$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n,$$
where $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ is the row $1, n, \ldots, n, 1$ in Pascal's Triangle:
```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

8. Rationalize the denominator: $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$
$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}} \cdot \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}^2 - \sqrt{b}^2} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

9. Complete the square to reduce a quadratic to only one $x$-variable:
$$ax^2+bx+c = a(x^2+\tfrac{b}{a}x)+c = a(x+\tfrac{b}{2a})^2 - a(\tfrac{b}{2a})^2 + c = a(x+\tfrac{b}{2a})^2 - \tfrac{b^2-4ac}{4a}.$$
Quadratic formula for roots: $ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

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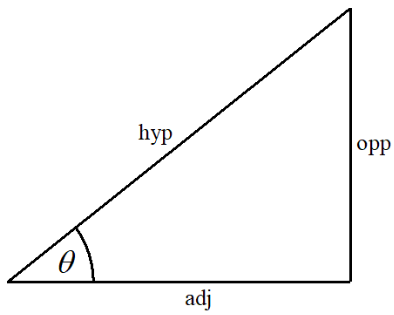
Notes by Peter Magyar `magyar@math.msu.edu`

**Trigonometry**

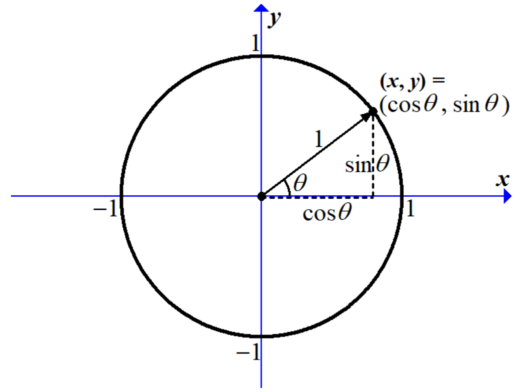
1. Right triangle definitions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$




2. Circular coordinates

$$\cos^2\theta + \sin^2\theta = 1$$




3. Quotient and co-functions

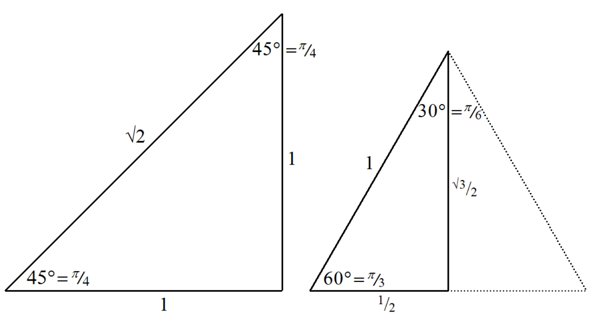
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \sin(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta) \qquad \csc\theta = \sec(90^\circ - \theta)$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

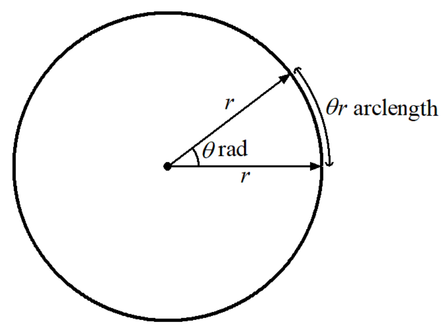
4. Special Triangles

isosceles right      half equilateral




5. Radian angle measure

$$360^\circ = 2\pi \text{ rad} \qquad 1^\circ = \tfrac{2\pi}{360} \text{ rad}$$



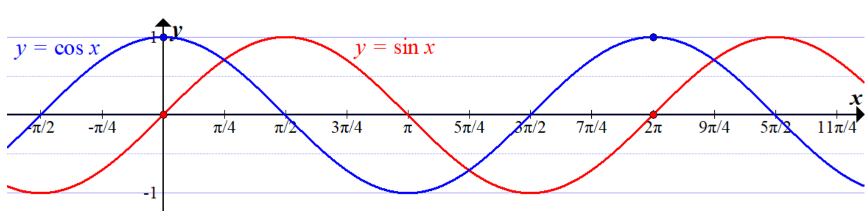

6. Angle addition formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

7. Half angle formulas $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$ $\qquad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

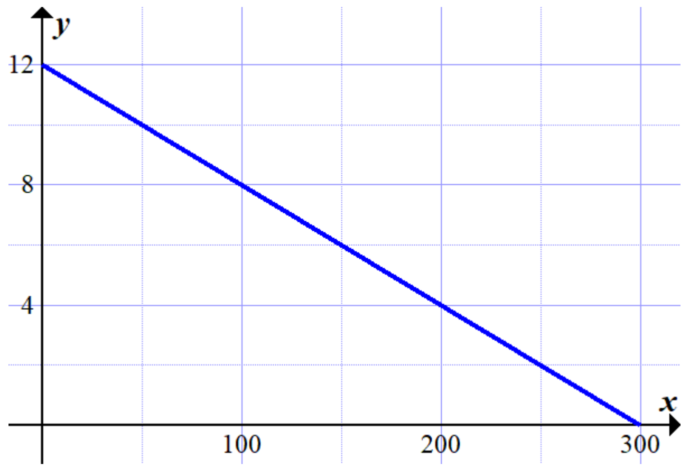
8. Graphs



**Linear function example: gallons per mile**

1. Physical: A gas tank has $y$ gallons after $x$ miles driven. Start at 12 gal, with constant gas usage rate 25 mi/gal, or $\frac{1}{25}$ gal/mi.
   Question: How much gasoline is left after driving 350 miles?

2. Graphical: Linear function $y = f(x)$, $y$-intercept $(x, y) = (0, 12)$, slope $-\frac{1}{25}$.




   The linear function model is valid only up to the $x$-intercept $(x, y) = (300, 0)$, so run out of gas at 300 mi!

3. Numerical: Input-output table of the function $y = f(x)$ at sample inputs $x$.

| $x$ mi | 0 | 50 | 100 | 150 | 200 | 250 | 300 |
|---|---|---|---|---|---|---|---|
| $y$ gal | 12 | 10 | 8 | 6 | 4 | 2 | 0 |

4. Algebraic: Intercept-slope formula $y = f(x) = 12 - \frac{1}{25}x$, so $f(350) = -2$.
   Formula only valid until $x$-intercept $12 - \frac{1}{25}x = 0$, i.e. $x = (12)(25) = 300$, but $f(350) = -2$ means you need 2 gal extra in the tank to make it 350 mi.

**Linear function example: miles per hour**

1. Physical: A car enters the highway at mile 50 at 1:00pm, and exits at mile 401 at 5:30pm. Question: What is the mile location at 2:30pm?

2. Graphical: Location is mile $s$ at $t$ o'clock. Linear function $s = f(t)$ between points $(t_1, s_1) = (1, 50)$ and $(t_2, s_2) = (5.5, 401)$.
   Position at $t = 2.5$ is approximately $s = f(2.5) \approx 170$.


3. Numerical: Input-output table of the function $s = f(t)$ at sample inputs $t$.
   speed $= \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{401-50}{5.5-1.0} = \frac{351 \text{ mi}}{4.5 \text{ hr}} = 78$ mi/hr, 39 mi in 0.5 hr.

| $t$ hr | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s$ mi | 50 | 89 | 128 | 167 | 206 | 245 | 284 | 323 | 362 | 401 |

   Get $s = f(2.5) = 167$.

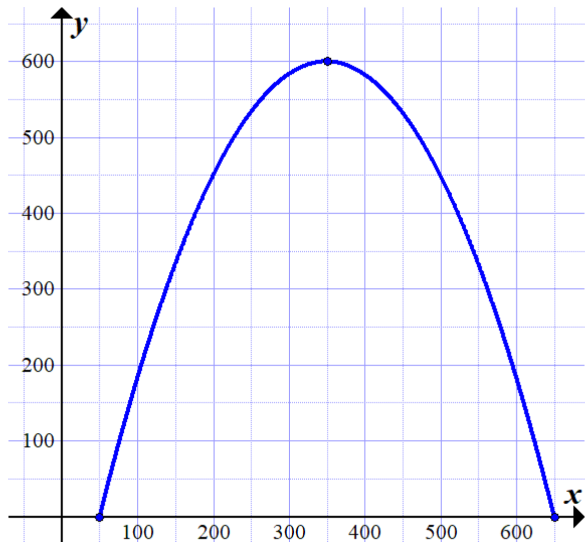
4. Algebraic: Slope between points $(t_1, s_1)$, $(t_2, s_2)$ is $m = \frac{s_2 - s_1}{t_2 - t_1} = 78$.
   Point-slope formula $f(t) = s_1 + m(t - t_1) = 50 + 78(t-1)$.
   Thus $f(2.5) = 50 + 78(2.5-1) = 167$.

**Quadratic function example: arch**

1. Physical: The Gateway Arch can be modeled by a downward-cupping parabola with width and height both equal to 600 ft.* One leg of the arch is positioned 50 ft from a base point, the other leg is at position 650 ft.

   Question: Find a formula $h(x)$ for the arch height above position $x$ ft.

2. Graphical: height $y = h(x)$




3. Algebraic: We have the roots $h(x) = 0$ for $x = 50$ and $x = 650$, so the factored form is $h(x) = a(x-50)(x-650)$ for some coefficient $a < 0$ (since the curve cups downward). The top of the arch is above the midpoint $x = \frac{50+650}{2} = 350$, so $h(350) = 600$, $a(350-50)(350-650) = 600$, $a = -\frac{1}{150}$.

$$h(x) \;=\; -\tfrac{1}{150}(x-50)(x-650) \;=\; -\tfrac{1}{150}x^2 + \tfrac{14}{3}x - \tfrac{650}{3}.$$

4. Numerical: Construction engineers use approximate $h(x)$ at sample $x$ values.

| $x$ ft | 50 | 100 | 150 | 200 | 250 | 300 | 350 | 400 | 450 | 500 | 550 | 600 | 650 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h(x)$ ft | 0 | 183 | 333 | 450 | 533 | 583 | 600 | 583 | 533 | 450 | 333 | 183 | 0 |

*A more accurate model is an inverted catenary or hyperbolic cosine curve (Calculus 2).

**Rational function example: concentration of a mixture**

1. Physical: A mixing tank contains a solution of 15 lb sugar in 100 gal water. The tank is then filled with 10 gal/min of water and 1 lb/min of sugar.

   Question: What is the sugar concentration after 10 min? What is the eventual concentration after a long time?

2. Numerical: Concentration function $c(t)$ lb/gal after $t$ min. Start with $c(0) = \frac{15 \text{ lb}}{100 \text{ gal}}$, then each minute, sugar increases by 1 lb and water by 10 gal.

| $t$ min | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(t)$ lb/gal | $\frac{15}{100}$ | $\frac{16}{110}$ | $\frac{17}{120}$ | $\frac{18}{130}$ | $\frac{19}{140}$ | $\frac{20}{150}$ | $\frac{21}{160}$ | $\frac{22}{170}$ | $\frac{23}{180}$ | $\frac{24}{190}$ | $\frac{25}{200}$ |
| | 0.150 | 0.145 | 0.141 | 0.138 | 0.135 | 0.133 | 0.131 | 0.129 | 0.127 | 0.126 | 0.125 |

   The concentration starts out at $c(0) = 0.15$ and decreases to $c(10) = 0.125$. Later, $c(100) = \frac{115}{1100}$, $c(1000) = \frac{1015}{10100}$, asymptotically approaching $\frac{1}{10} = 0.1$ .

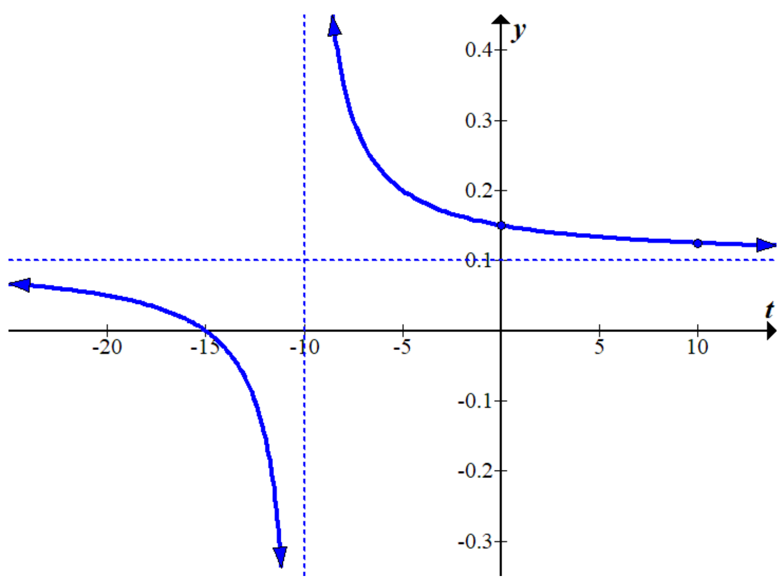
3. Algebraic: The amount of sugar is $15 + t$, the amount of water $100 + 10t$, so the concentration is $c(t) = \frac{15+t}{100+10t}$. For $t < 0$, this gives the concentration in the past, assuming the same fill rates. This makes no sense past where the denominator is zero, $t = -10$: at this time, the tank started with 5 lb sugar and no water, an infinite concentration: $c(-10) = \infty$, or more precisely, $c(t) \to \infty$ as $t \to -10$, where the arrow $\to$ means "approaches". For $t < -10$, the formula has no physical meaning.

   Polynomial long division gives $(t{+}15) \div (10t{+}100) = \frac{1}{10}$ remainder 5, so:

$$c(t) = \frac{t+15}{10t+100} = \frac{1}{10} + \frac{5}{10t+100}.$$

   For very large $t \to \infty$, the second term goes to 0, so $c(t) \to \frac{1}{10}$.

4. Graphical: Plot the curve $y = c(t)$. The line where the denominator $= 0$ is a vertical asymptote $t = -10$, and the limiting value as $t \to \infty$ is a horizontal asymptote $y = \frac{1}{10}$. The graph is a hyperbola curve.



**Periodic function example: hours of daylight**

1. Physical: In Michigan on Jun 21, the longest day of the year, there is about $15\frac{1}{4}$ hours of daylight; and on Dec 21, the shortest day, there is about 9 hours. How long a day would we expect on Aug 1?

2. Numerical: Measuring time by days of the year (ignoring leap day), Jun 21 is day 171, and Dec 21 is day 354. Measuring daylight by hours, the maximum of the daylight function $\ell(t)$ is $\ell(171) = 15.25$; its minimum is $\ell(354) = 9$.

   Aug 1 is day 212, which is $\frac{212-171}{354-171} = 0.22$ of the way from the maximum to the minimum day, so linear interpolation suggests that $\ell(212)$ is lower than the maximum by 0.22 of the max-to-min daylight difference $15.25-9 = 6.25$:

   $$\ell(212) \;\approx\; 15.25 - \tfrac{212-171}{354-171}(15.25-9) \;=\; 15.25 - (0.22)(6.25) \;=\; 13.9 \text{ hr}\,.$$

   However, we expect this to be too low, because the daylight decreases only slowly after its Jun 21 solstice, and accelerates only near the Sep 21 equinox.

3. Algebraic. We model the undulations of daylight by a sinusoidal function:

   $$\ell(t) \;=\; a + b\sin(ct+d).$$

   We must adjust the constants $a, b, c, d$ so that the properties of the sine function are scaled to the desired properties of $\ell(t)$.

   Sine has max/min values 1 and $-1$, whereas $\ell$ has 15.25 and 9; thus we should have $a + b = 15.25$ and $a - b = 9$, giving:

   $$a \;=\; \tfrac{15.25+9}{2} \;=\; 12.12\,, \qquad\qquad b \;=\; \tfrac{15.25-9}{2} \;=\; 3.12\,.$$

   Sine has period $2\pi$, while $\ell$ has period 365, so $2\pi = 365c$. The function $\sin x$ has a maximum at $x = \frac{\pi}{2}$, while $\ell(t)$ has $t = 171$, so we want $\frac{\pi}{2} = 171c + d$:

   $$c \;=\; \tfrac{2\pi}{365} \;=\; 0.017\,, \qquad d \;=\; \tfrac{\pi}{2} - 171c \;=\; \tfrac{\pi}{2} - (171)\tfrac{2\pi}{365} \;=\; -1.37\,.$$

   Therefore, our model is: $\ell(t) \;=\; 12.12 + 3.12\sin(0.017\,t - 1.37)$.

   For Aug 1 = day 212, this predicts $\ell(212) = 14.6 = 14\,\text{hr}\ 36\,\text{min}$ of daylight.

   Observations give the actual time as $14.43 = 14\,\text{hr}\ 26\,\text{min}$, showing that the true daylight function is a little flatter than a sinusoid, but this is still much more accurate than the piecewise-linear model.

4. Graphical: Plot the piecewise-linear and sinusoidal model curves.