Region between two parabolas. We have seen that geometrically, the integral \( \int_a^b f(x) \, dx \) computes the area between a curve \( y = f(x) \) and an interval \( x \in [a, b] \) on the \( x \)-axis (with area below the axis counted negatively). In Calculus II, we will show the versatility of the integral to compute all kinds of areas, lengths, volumes: almost any measure of size for a geometric object.

In this section, we compute more general areas: those between two given curves \( y = f(x) \) and \( y = g(x) \), usually with no boundary on the \( x \)-axis.

Example: Consider the region with top boundary \( y = f(x) = x^2+x+1 \), bottom boundary \( y = g(x) = 2x^2-1 \), left boundary the \( y \)-axis \( x = 0 \), right boundary \( x = 1 \):†

\[
R = \{ (x, y) \text{ with } g(x) \leq y \leq f(x) \text{ and } x \in [0, 1] \}.
\]

Here \( y = g(x) = 2x^2-1 \) is a standard parabola shifted downward, with minimum point \( x = 0 \). The curve \( y = f(x) = x^2+x+1 \) is roughly like its leading term \( y = x^2 \), a parabola opening upward; its minimum point satisfies \((x^2+x+1)' = 2x+1 = 0\), i.e. \( x = -\frac{1}{2} \).

To compute the area of \( R \), we use the same geometric-numerical strategy as for the region under a single curve: split \( R \) into \( n \) thin vertical slices of width \( \Delta x = \frac{1}{n} \), each approximately a rectangle; then add up the rectangle areas and take the limit as \( n \) becomes larger and larger. In the interval \( x \in [0, 1] \), we take sample points \( x_1, \ldots, x_n \), one in each \( \Delta x \) increment. The slice at position \( x_i \) has height equal to the ceiling minus the floor, \( f(x_i) - g(x_i) \), so:

\[
\text{area of slice } \approx \text{(height)} \times \text{(width)} = (f(x_i) - g(x_i)) \Delta x,
\]

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†We specify the region as the set of all points \((x, y)\) which satisfy the given conditions.
and the total area is:

\[
A_{[0,1]} = \lim_{n \to \infty} \sum_{i=1}^{n} (f(x_i) - g(x_i)) \Delta x = \int_{0}^{1} (f(x) - g(x)) \, dx
\]

\[
= \int_{0}^{1} (x^2 + x + 1) - (2x^2 - 1) \, dx = \left[ 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{x=0}^{x=1} = \frac{13}{6}.
\]

**EXAMPLE:** Next, consider the region between the same curves \( y = f(x) = x^2 + x + 1 \) and \( y = g(x) = 2x^2 - 1 \), but above the interval \( x \in [1, 3] \). To picture the region without a calculator, we determine the intersection points where the curves cross:

\[
f(x) = g(x) \iff x^2 + x + 1 = 2x^2 - 1 \iff x^2 - x - 2 = 0 \iff x = \frac{1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm 3}{2} = -1 \text{ or } 2
\]

by the Quadratic Formula. Only \( x = 2 \) is relevant for our region above \( x \in [1, 3] \).

At \( x = 1 \) we have \( g(1) < f(1) \), so to the left of \( x = 2 \), our region is defined by \( g(x) \leq y \leq f(x) \). At \( x = 3 \), we have \( f(3) < g(3) \), so to the right of \( x = 2 \), it is \( f(x) \leq y \leq g(x) \):

Repeating our previous area formula for the two parts of our region gives:

\[
A_{[1,3]} = A_{[1,2]} + A_{[2,3]} = \int_{1}^{2} (f(x) - g(x)) \, dx + \int_{2}^{3} (g(x) - f(x)) \, dx
\]

\[
= \int_{1}^{2} (2 + x - x^2) \, dx + \int_{2}^{3} (-2 - x + x^2) \, dx = \frac{7}{6} + \frac{11}{6} = 3.
\]
EXAMPLE: Finally, we consider the same curves \( y = f(x) = x^2 + x + 1 \) and \( y = g(x) = 2x^2 - 1 \), but we take the entire finite region between them:

\[
R' = \{ (x, y) \text{ with } g(x) \leq y \leq f(x) \}.
\]

Here the top boundary is \( y = x^2 + x + 1 \) and the bottom boundary is \( y = 2x^2 - 1 \), but we have not specified an \( x \) interval. However, we have already computed the intersection points \( x = -1 \) and \( x = 2 \), and the curves do not enclose any finite regions beyond these points. Thus:

\[
A_{[-1,2]} = \int_{-1}^{2} (f(x) - g(x)) \, dx = \frac{9}{2}.
\]

We can generalize the above examples in:

**Theorem:** The area of the region enclosed between \( f(x) \) and \( g(x) \) for \( x \in [a, b] \) is: \( A = \int_{a}^{b} |f(x) - g(x)| \, dx \).

The absolute value signs ensure we take the integral of top minus bottom, regardless of which is which. In practice, we must find the intersection points where \( f(x) = g(x) \), which split the integral into intervals where \( g(x) \leq f(x) \) versus \( f(x) \leq g(x) \).

**Integrating with respect to \( y \).** Consider the region:

\[
R = \{ (x, y) \text{ with } y^2 \leq x \leq y + 1 \}.
\]

Here the boundary curves are naturally graphs in which \( y \) is the independent variable: the right boundary is the line \( x = f(y) = y + 1 \); and the left boundary is \( x = g(y) = y^2 \), a parabola opening to the right.
Understand: it is merely by habit that we consider $y$ as a function of $x$. We can make $x$ a function of $y$ instead if it is more convenient, and the same formulas will work if we switch the roles of $x$ and $y$. Thus, we find the intersection points:

$$y + 1 = y^2$$

when $y = \frac{1 \pm \sqrt{5}}{2}$ by the Quadratic Formula. The area as:

$$A = \int_{1 - \sqrt{2}}^{1 + \sqrt{2}} (y + 1) - y^2 \, dy = \left[ \frac{1}{2} y^2 + y - \frac{1}{2} y^3 \right]_{y=1 - \sqrt{2}}^{y=1 + \sqrt{2}} = \frac{5}{6} \sqrt{5}.$$

Here $((y+1) - (y^2)) \, dy$ represents the area of the horizontal slice of the region at height $y$, with thickness $dy$.

To check this, we re-do it from our usual perspective, using $x$ as the independent variable. This makes it more complicated, since we must consider the region as having three boundary graphs: upper boundary $y = \sqrt{x}$, lower right boundary $y = x - 1$, and lower left boundary $y = -\sqrt{x}$. The intersection points are:

- Between $y = \sqrt{x}$ and $y = x - 1$: $x = \frac{3 + \sqrt{5}}{2}$ (upper right corner)
- Between $y = -\sqrt{x}$ and $y = x - 1$: $x = \frac{3 - \sqrt{5}}{2}$ (lower middle corner)
- Between $y = \sqrt{x}$ and $y = -\sqrt{x}$: $x = 0$ (left end)

These split the region into left and right parts:

The area is:

$$A = \int_{0}^{\frac{3 - \sqrt{5}}{2}} (\sqrt{x}) - (-\sqrt{x}) \, dx + \int_{\frac{3 - \sqrt{5}}{2}}^{\frac{3 + \sqrt{5}}{2}} (\sqrt{x}) - (x-1) \, dx,$$

which after much algebra gives the same answer as before: $\frac{5}{6} \sqrt{5}$. 