Reversing the Chain Rule. As we have seen from the Second Fundamental Theorem (§4.3), the easiest way to evaluate an integral $\int_a^b f(x)\,dx$ is to find an antiderivative, the indefinite integral $\int f(x)\,dx = F(x) + C$, so that $\int_a^b f(x)\,dx = F(b) - F(a)$. Building on §3.9, we will find antiderivatives by reversing our methods of differentiation: here, we reverse the Chain Rule, $F(g(x))' = F'(g(x))\,g'(x)$.

For example, let us find the antiderivative:

$$\int x\cos(x^2)\,dx\,.$$

That is, for what function will the Derivative Rules produce $x\cos(x^2)$? We notice an inside function $g(x) = x^2$, and a factor x which is very close to the derivative $g'(x) = 2x$. In fact, we can get the exact derivative of the inside function if we multiply the factors by $\frac{1}{2}$ and 2:

$$x\cos(x^2) \;=\; \tfrac{1}{2}\cos(x^2)\cdot(2x) \;=\; \tfrac{1}{2}\cos(x^2)\cdot(x^2)'\,.$$

This is just the kind of derivative function produced by the Chain Rule:

$$F(g(x))' \;=\; F'(g(x))\cdot g'(x) \;=\; F'(x^2)\cdot(x^2)' \;\overset{??}{=}\; \tfrac{1}{2}\cos(x^2)\cdot(2x)\,.$$

We still need to find the outside function F. To remind us of the original inside function, we write $F(u)$, where the new variable u represents $u = g(x) = x^2$. We must get $F'(u) = \frac{1}{2}\cos(u)$, an easy antiderivative:

$$\int \tfrac{1}{2}\cos(u)\,du \;=\; F(u) + C \;=\; \tfrac{1}{2}\sin(u) + C\,.$$

Now we restore the original inside function to get our final answer:

$$\int \tfrac{1}{2}\cos(u)\,du \;=\; \tfrac{1}{2}\sin(u) + C \;=\; \tfrac{1}{2}\sin(x^2) + C\,.$$

The Chain Rule in Leibnitz notation (§2.5) reverses and checks the above computation. Writing $y = \frac{1}{2}\sin(u)$ and $u = x^2$:

$$\frac{dy}{dx} \;=\; \frac{dy}{du}\cdot\frac{du}{dx} \;=\; \frac{d}{du}\big(\tfrac{1}{2}\sin(u)\big)\cdot\frac{d}{dx}\big(x^2\big)$$
$$=\; \tfrac{1}{2}\cos(u)\cdot(2x) \;=\; \tfrac{1}{2}\cos(x^2)\cdot(2x) \;=\; x\cos(x^2)\,.$$

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Substitution Method

1. Given an antiderivative $\int h(x)\,dx$, try to find an inside function $g(x)$ such that $g'(x)$ is a factor of the integrand:
$$h(x) \;=\; f(g(x))\cdot g'(x).$$
This will often involve multiplying and dividing by a constant to get the exact derivative $g'(x)$. After factoring out $g'(x)$, sometimes the remaining factor needs to be manipulated to write it as a function of $u=g(x)$.

2. Using the symbolic notation $u=g(x),\;\; du=\dfrac{du}{dx}\,dx=g'(x)\,dx$, write:
$$\int h(x)\,dx \;=\; \int f(g(x))\cdot g'(x)\,dx \;=\; \int f(u)\,du\,,$$
and find the antiderivative $\int f(u)\,du=F(u)+C$ by whatever method.

3. Restore the original inside function:
$$\int h(x)\,dx \;=\; \int f(u)\,du \;=\; F(u)+C \;=\; F(g(x))+C\,.$$

Examples

- $\int (3x{+}4)\sqrt{3x{+}4}\,dx$. The inside function is clearly $u=3x{+}4$, $du=3\,dx$, so:
$$\int (3x{+}4)\sqrt{3x{+}4}\,dx \;=\; \int \tfrac{1}{3}(3x{+}4)\sqrt{3x{+}4}\cdot 3\,dx$$
$$=\; \int \tfrac{1}{3}u\sqrt{u}\,du \;=\; \tfrac{1}{3}\int u^{3/2}\,du \;=\; \tfrac{1}{3}\,\tfrac{2}{5}u^{5/2}+C \;=\; \tfrac{2}{15}(3x{+}4)^{5/2}+C.$$

- $\displaystyle\int x\sqrt{3x{+}4}\,dx$. Again $u=3x{+}4$, so $\sqrt{3x{+}4}$ becomes $\sqrt{u}$, but we must still express the remaining factor x in terms of u. We solve $u=3x{+}4$ to obtain $x=\frac{1}{3}u-\frac{4}{3}$: that is, $x=\frac{1}{3}(3x{+}4)-\frac{4}{3}$:
$$\int x\sqrt{3x{+}4}\,dx \;=\; \int \tfrac{1}{3}\big(\tfrac{1}{3}(3x{+}4){-}\tfrac{4}{3}\big)\sqrt{3x{+}4}\cdot 3\,dx \;=\; \int \tfrac{1}{3}\big(\tfrac{1}{3}u{-}\tfrac{4}{3}\big)\sqrt{u}\,du$$
$$=\; \int \tfrac{1}{9}u^{3/2}{-}\tfrac{4}{9}u^{1/2}\,du \;=\; \tfrac{1}{9}\,\tfrac{2}{5}u^{5/2}{-}\tfrac{4}{9}\,\tfrac{2}{3}u^{3/2}{+}C \;=\; \tfrac{2}{45}(3x{+}4)^{5/2}{-}\tfrac{8}{27}(3x{+}4)^{3/2}{+}C.$$

- $\displaystyle\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx\,.$ We take $u=\sqrt{x}=x^{1/2}$, $du=\frac{1}{2}x^{-1/2}\,dx=\frac{1}{2\sqrt{x}}\,dx$
$$\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx \;=\; \int 2\sec^2(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\,dx$$
$$=\; \int \sec^2(u)\,du \;=\; \tan(u)+C \;=\; \tan(\sqrt{x})+C.$$
Here we use the trig integrals from §3.9.

- $\displaystyle\int \frac{\sin(x)}{(1+\cos(x))^2}\,dx$. We cannot take the inside function $u=\sin(x)$, because its derivative $\cos(x)$ is not a factor of the integrand. We could take $u = \cos(x)$, but the best choice is $u = 1+\cos(x)$, $du = -\sin(x)\,dx$:

$$\int \frac{\sin(x)}{(1+\cos(x))^2}\,dx \;=\; -\int \frac{1}{(1+\cos(x))^2}\cdot(-\sin(x))\,dx$$

$$=\; -\int \frac{1}{u^2}\,du \;=\; \frac{1}{u}+C \;=\; \frac{1}{1+\cos(x)}+C.$$

- $\displaystyle\int \frac{1-\sqrt{x}}{\sqrt{1+\sqrt{x}}}\,dx$. Take $u = 1+\sqrt{x}$, $du = \frac{1}{2\sqrt{x}}$, so $\sqrt{x} = u-1,\;\; 1-\sqrt{x} = 2-u$.

$$\int \frac{1-\sqrt{x}}{\sqrt{1+\sqrt{x}}}\,dx \;=\; \int \frac{1-\sqrt{x}}{\sqrt{1+\sqrt{x}}}\,(2\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$=\; \int \frac{2-u}{\sqrt{u}}\,2(u-1)\,du \;=\; -2\int \frac{u^2-3u+2}{\sqrt{u}}\,du$$

$$=\; -2\int u^{3/2}-3u^{1/2}+2u^{-1/2}\,du \;=\; -2(\tfrac{2}{5}u^{5/2}-2u^{3/2}+4u^{1/2})+C$$

$$=\; -\tfrac{4}{5}(1+\sqrt{x})^{5/2}+4(1+\sqrt{x})^{3/2}-8(1+\sqrt{x})^{1/2}+C.$$

Whew! Here we did not have the derivative factor $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ already present: we had to multipy and divide by it to get du, then express the remaining factors in terms of u. By luck, the resulting $\int f(u)\,du$ was do-able.

- $\displaystyle\int \sec^2(x)\tan(x)\,dx$. Here we could take $u = \tan(x)$, $du = \sec^2(x)\,dx$:

$$\int \sec^2(x)\tan(x)\,dx \;=\; \int \tan(x)\cdot\sec^2(x)\,dx$$

$$=\; \int u\,du \;=\; \tfrac{1}{2}u^2+C \;=\; \tfrac{1}{2}\tan^2(x)+C.$$

Alternatively, use the inside function $z=\sec(x)$, $dz = \tan(x)\sec(x)\,dx$:

$$\int \sec^2(x)\tan(x)\,dx \;=\; \int \sec(x)\cdot\tan(x)\sec(x)\,dx$$

$$=\; \int z\,dz \;=\; \tfrac{1}{2}z^2+C \;=\; \tfrac{1}{2}\sec^2(x)+C.$$

Thus $\frac{1}{2}\tan^2(x)$ and $\frac{1}{2}\sec^2(x)$ are two different antiderivatives, but what about the Antiderivative Uniqueness Theorem (§3.9)? In fact, the identity $\tan^2(x)+1 = \sec^2(x)$ implies:

$$\tfrac{1}{2}\tan^2(x)+\tfrac{1}{2} \;=\; \tfrac{1}{2}\sec^2(x)\,.$$

These give the *same* antiderivative family: $\frac{1}{2}\tan^2(x)+C \;=\; \frac{1}{2}\sec^2(x)+C'\,!$

Substitution for definite integrals. We have, for $u = g(x)$:

$$\int_a^b f(g(x))\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du\,.$$

EXAMPLE: $\int_2^3 x(1{+}x^2)^5\, dx$. Taking $u = 1{+}x^2$, $du = 2x\, dx$:

$$\int_3^4 x(1{+}x^2)^5\, dx \;=\; \int_3^4 \tfrac{1}{2}(1{+}x^2)^5 \cdot 2x\, dx \;=\; \int_{1+3^2}^{1+4^2} \tfrac{1}{2}u^5\, du$$

$$=\; \tfrac{1}{12}\, u^6 \,\Big|_{u=10}^{u=17} \;=\; \tfrac{1}{12}10^6 - \tfrac{1}{12}17^6\,.$$

Integral Symmetry Theorem: If $f(x)$ is an odd function, meaning $f(-x) = -f(x)$, then $\int_{-a}^a f(x)\, dx = 0$.

Proof. By the Integral Splitting Rule (§4.2), we have:

$$\int_{-a}^a f(x)\, dx \;=\; \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx\,.$$

Substituting $u = -x$, $du = (-1)\, dx$ in the first term, including in the limits of integration, and using $f(-x) = -f(x)$, we get:

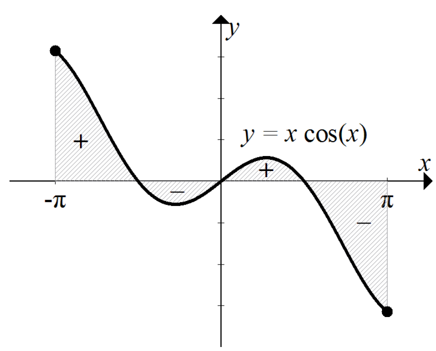
$$\int_{-a}^0 f(x)\, dx \;=\; \int_{-a}^0 -f(x)\cdot(-1)\, dx \;=\; \int_{-a}^0 f(-x)\cdot(-1)\, dx$$

$$=\; \int_{-(-a)}^{-0} f(u)\, du \;=\; \int_a^0 f(u)\, du \;=\; -\int_0^a f(u)\, du \;=\; -\int_0^a f(x)\, dx\,.$$

The last equality holds because the variable of integration is merely suggestive, and can be changed arbitrarily. Therefore $\int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx = -\int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 0$, as desired.

EXAMPLE: Evaluate the definite integral $\int_{-\pi}^{\pi} x\cos(x)\, dx$. Here substitution will not work, and it is difficult to find an antiderivative. But since $(-x)\cos(-x) = -(x\cos(x))$, the Theorem tells us the integral must be *zero*.

Geometrically, the integral is the signed area between the graph and the x-axis:



Since the function $f(x) = x\cos(x)$ is odd, the graph has rotational symmetry around the origin, and each negative area below the x-axis cancels a positive area above the x-axis.

Application: Heart Flow Rate. In a standard medical test to measure the rate of blood pumped by the heart, r liters/min, doctors inject a colored dye into a vein flowing toward the heart, then measure the concentration of dye in arterial blood as it is pumped out from the heart, $c(t)$ mg/liter after t minutes.

PROBLEM: Given the dye concentration function $c(t)$, determine the flow rate r.

Let the variable ℓ denote the volume of blood which has flowed through the artery since the start time. Assuming the (unknown) flow rate r is constant, we have $\ell = rt$. Let us define $C(\ell)$ to be the dye concentration after ℓ liters have flowed, so that $C(\ell) = C(rt) = c(t)$.

Now, the integral $\int_0^\infty C(\ell)\,d\ell$, summing up concentration (mg/liter) times increments of volume (liters), measures the total amount D of dye (mg):

$$D \;=\; \int_0^\infty C(\ell)\,d\ell.$$

Performing the substitution $\ell = rt$, $d\ell = r\,dt$, we have:

$$D \;=\; \int_0^\infty C(rt)\,r\,dt \;=\; r\int_0^\infty c(t)\,dt.$$

Then we may compute r as:

$$r \;=\; \frac{D}{\int_0^\infty c(t)\,dt}.$$

Since the total dye D is known (the amount injected), $c(t)$ is measured by the test, and $\int_0^\infty c(t)\,dt$ can be computed by Riemann sums,* we obtain flow rate r.

* Since $c(t) = 0$ after all the dye has passed, $\int_0^\infty c(t)\,dt$ can be cut off to a finite integral.