Reversing the Chain Rule. As we have seen from the Second Fundamental Theorem (§4.3), the easiest way to evaluate an integral \( \int_a^b f(x) \, dx \) is to find an antiderivative, the indefinite integral \( \int f(x) \, dx = F(x) + C \), so that \( \int_a^b f(x) \, dx = F(b) - F(a) \). Building on §3.9, we will find antiderivatives by reversing our methods of differentiation: here, we reverse the Chain Rule, \( F(g(x))' = F'(g(x)) \cdot g'(x) \).

For example, let us find the antiderivative:

\[
\int x \cos(x^2) \, dx.
\]

That is, for what function will the Derivative Rules produce \( x \cos(x^2) \)? We notice an inside function \( g(x) = x^2 \), and a factor \( x \) which is very close to the derivative \( g'(x) = 2x \). In fact, we can get the exact derivative of the inside function if we multiply the factors by \( \frac{1}{2} \) and 2:

\[
x \cos(x^2) = \frac{1}{2} \cos(x^2) \cdot (2x) = \frac{1}{2} \cos(x^2) \cdot (x^2)'.
\]

This is just the kind of derivative function produced by the Chain Rule:

\[
F(g(x))' = F'(g(x)) \cdot g'(x) = F'(x^2) \cdot (x^2)' = \frac{1}{2} \cos(x^2) \cdot (2x).
\]

We still need to find the outside function \( F \). To remind us of the original inside function, we write \( F(u) \), where the new variable \( u \) represents \( u = g(x) = x^2 \). We must get \( F'(u) = \frac{1}{2} \cos(u) \), an easy antiderivative:

\[
\int \frac{1}{2} \cos(u) \, du = F(u) + C = \frac{1}{2} \sin(u) + C.
\]

Now we restore the original inside function to get our final answer:

\[
\int \frac{1}{2} \cos(u) \, du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(x^2) + C.
\]

The Chain Rule in Leibnitz notation (§2.5) reverses and checks the above computation. Writing \( y = \frac{1}{2} \sin(u) \) and \( u = x^2 \):

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} \left( \frac{1}{2} \sin(u) \right) \cdot \frac{d}{dx} (x^2)
\]

\[
= \frac{1}{2} \cos(u) \cdot (2x) = \frac{1}{2} \cos(x^2) \cdot (2x) = x \cos(x^2).
\]
Substitution Method

1. Given an antiderivative \( \int h(x) \, dx \), try to find an inside function \( g(x) \) such that \( g'(x) \) is a factor of the integrand:

\[
h(x) = f(g(x)) \cdot g'(x).
\]

This will often involve multiplying and dividing by a constant to get the exact derivative \( g'(x) \). After factoring out \( g'(x) \), sometimes the remaining factor needs to be manipulated to write it as a function of \( u = g(x) \).

2. Using the symbolic notation \( u = g(x) \), \( du = \frac{du}{dx} \, dx = g'(x) \, dx \), write:

\[
\int h(x) \, dx = \int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du,
\]

and find the antiderivative \( \int f(u) \, du = F(u) + C \) by whatever method.

3. Restore the original inside function:

\[
\int h(x) \, dx = \int f(u) \, du = F(u) + C = F(g(x)) + C.
\]

Examples

- \( \int (3x+4)\sqrt{3x+4} \, dx \). The inside function is clearly \( u = 3x+4 \), \( du = 3 \, dx \), so:

\[
\int (3x+4)\sqrt{3x+4} \, dx = \int \frac{1}{3}(3x+4)\sqrt{3x+4} \cdot 3 \, dx \\
= \int \frac{1}{3}u^{\frac{3}{2}} \, du = \frac{1}{6} \int u^{\frac{3}{2}} \, du = \frac{1}{6} \cdot \frac{2}{5}u^{\frac{5}{2}} + C = \frac{2}{15}(3x+4)^{\frac{5}{2}} + C.
\]

- \( \int x\sqrt{3x+4} \, dx \). Again \( u = 3x+4 \), so \( \sqrt{3x+4} \) becomes \( \sqrt{u} \), but we must still express the remaining factor \( x \) in terms of \( u \). We solve \( u = 3x+4 \) to obtain \( x = \frac{1}{3}u - \frac{4}{3} \); that is, \( x = \frac{1}{3}(3x+4) - \frac{4}{3} \):

\[
\int x\sqrt{3x+4} \, dx = \int \frac{1}{3}(\frac{1}{3}(3x+4)-\frac{4}{3})\sqrt{3x+4} \cdot 3 \, dx = \int \frac{1}{3}(\frac{1}{3}u-\frac{4}{3})\sqrt{u} \, du \\
= \int \frac{1}{9}u^{\frac{3}{2}} - \frac{4}{9}u^{\frac{1}{2}} \, du = \frac{1}{9} \cdot \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{9} \cdot \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{15}(3x+4)^{\frac{5}{2}} - \frac{8}{27}(3x+4)^{\frac{3}{2}} + C.
\]

- \( \int \sec^2(\sqrt{x}) \, \sqrt{x} \) \, dx \). We take \( u = \sqrt{x} = x^{1/2} \), \( du = \frac{1}{2}x^{-1/2} \, dx = \frac{1}{2\sqrt{x}} \, dx \):

\[
\int \sec^2(\sqrt{x}) \, \sqrt{x} \, dx = \int 2\sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \, dx \\
= \int \sec^2(u) \, du = \tan(u) + C = \tan(\sqrt{x}) + C.
\]

Here we use the trig integrals from §3.9.
• $\int \frac{\sin(x)}{(1 + \cos(x))^2} \, dx$. We cannot take the inside function $u = \sin(x)$, because its derivative $\cos(x)$ is not a factor of the integrand. We could take $u = \cos(x)$, but the best choice is $u = 1 + \cos(x)$, $du = -\sin(x) \, dx$:

$$\int \frac{\sin(x)}{(1 + \cos(x))^2} \, dx = - \int \frac{1}{(1 + \cos(x))^2} \cdot (-\sin(x)) \, dx$$

$$= - \int \frac{1}{u^2} \, du = \frac{1}{u} + C = \frac{1}{1 + \cos(x)} + C.$$

• $\int \frac{1 - \sqrt{x}}{\sqrt{1 + \sqrt{x}}} \, dx$. Take $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}$, so $\sqrt{x} = u - 1$, $1 - \sqrt{x} = 2 - u$.

$$\int \frac{1 - \sqrt{x}}{\sqrt{1 + \sqrt{x}}} \, dx = \int \frac{1 - \sqrt{x}}{\sqrt{1 + \sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} \, dx$$

$$= \int \frac{2 - u}{\sqrt{u}} \cdot (2u) \, du = -2 \int \frac{u^2 - 3u + 2}{\sqrt{u}} \, du$$

$$= -2 \int u^{3/2} - 3u^{1/2} + 2u^{-1/2} \, du = -2\left(\frac{2}{5}u^{5/2} - 2u^{3/2} + 4u^{1/2}\right) + C$$

$$= -\frac{4}{5}(1 + \sqrt{x})^{5/2} + 4(1 + \sqrt{x})^{3/2} - 8(1 + \sqrt{x})^{1/2} + C.$$

Whew! Here we did not have the derivative factor $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ already present: we had to multiply and divide by it to get $du$, then express the remaining factors in terms of $u$. By luck, the resulting $\int f(u) \, du$ was do-able.

• $\int \sec^2(x) \tan(x) \, dx$. Here we could take $u = \tan(x)$, $du = \sec^2(x) \, dx$:

$$\int \sec^2(x) \tan(x) \, dx = \int \tan(x) \cdot \sec^2(x) \, dx$$

$$= \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2} \tan^2(x) + C.$$

Alternatively, use the inside function $z = \sec(x)$, $dz = \tan(x) \sec(x) \, dx$:

$$\int \sec^2(x) \tan(x) \, dx = \int \sec(x) \cdot \tan(x) \sec(x) \, dx$$

$$= \int z \, dz = \frac{1}{2}z^2 + C = \frac{1}{2} \sec^2(x) + C.$$

Thus $\frac{1}{2}\tan^2(x)$ and $\frac{1}{2}\sec^2(x)$ are two different antiderivatives, but what about the Antiderivative Uniqueness Theorem (§3.9)? In fact, the identity $\tan^2(x) + 1 = \sec^2(x)$ implies:

$$\frac{1}{2}\tan^2(x) + \frac{1}{2} = \frac{1}{2}\sec^2(x).$$

These give the same antiderivative family: $\frac{1}{2}\tan^2(x) + C = \frac{1}{2}\sec^2(x) + C'$. ！
Substitution for definite integrals. We have, for \( u = g(x) \):

\[
\int_{a}^{b} f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

**Example:** \( \int_{2}^{3} x(1+x^2)^5 \, dx \). Taking \( u = 1+x^2 \), \( du = 2x \, dx \):

\[
\int_{2}^{3} x(1+x^2)^5 \, dx = \int_{3}^{4} \frac{1}{2}(1+x^2)^5 \cdot 2x \, dx = \int_{1+3}^{1+4^2} \frac{1}{2}u^5 \, du = \frac{1}{12} u^6 \bigg|_{u=17}^{u=10} = \frac{1}{12} 10^6 - \frac{1}{12} 17^6.
\]

**Integral Symmetry Theorem:** If \( f(x) \) is an odd function, meaning \( f(-x) = -f(x) \), then \( \int_{-a}^{a} f(x) \, dx = 0 \).

**Proof.** By the Integral Splitting Rule (§4.2), we have:

\[
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx.
\]

Substituting \( u = -x \), \( du = (-1) \, dx \) in the first term, including in the limits of integration, and using \( f(-x) = -f(x) \), we get:

\[
\int_{-a}^{0} f(x) \, dx = \int_{-a}^{0} -f(x) \cdot (-1) \, dx = \int_{-a}^{0} f(-x) \cdot (-1) \, dx
\]

\[
= \int_{-(-a)}^{0} f(u) \, du = \int_{a}^{0} f(u) \, du = -\int_{0}^{a} f(u) \, du = -\int_{0}^{a} f(x) \, dx.
\]

The last equality holds because the variable of integration is merely suggestive, and can be changed arbitrarily. Therefore \( \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = -\int_{0}^{a} f(x) \, dx + \int_{0}^{a} f(x) \, dx = 0 \), as desired.

**Example:** Evaluate the definite integral \( \int_{-\pi}^{\pi} x \cos(x) \, dx \). Here substitution will not work, and it is difficult to find an antiderivative. But since \( (-x) \cos(-x) = -(x \cos(x)) \), the Theorem tells us the integral must be zero.

Geometrically, the integral is the signed area between the graph and the \( x \)-axis:
**Application: Heart Flow Rate.** In a standard medical test to measure the rate of blood pumped by the heart, \( r \) liters/min, doctors inject a colored dye into a vein flowing toward the heart, then measure the concentration of dye in arterial blood as it is pumped out from the heart, \( c(t) \) mg/liter after \( t \) minutes.

**Problem:** Given the dye concentration function \( c(t) \), determine the flow rate \( r \).

Let the variable \( \ell \) denote the liters of blood which have flowed through the artery since the start time. Assuming the (unknown) flow rate \( r \) is constant, we have \( \ell = rt \). Let \( C(\ell) \) be the dye concentration after \( \ell \) liters have flowed, so that \( C(\ell) = C(rt) = c(t) \).

Now, the integral \( \int_0^\infty C(\ell) \, d\ell \) sums up:

\[
(mg/\text{litrer concentration}) \times (\text{litrer increments}) = (mg \text{ increments of dye}),
\]

which computes the total amount of dye, \( D \) mg:

\[
D = \int_0^\infty C(\ell) \, d\ell.
\]

Performing the substitution \( \ell = rt, \, d\ell = r \, dt \), we have:

\[
D = \int_0^\infty C(rt) \, r \, dt = r \int_0^\infty c(t) \, dt.
\]

Then we may compute \( r \) as:

\[
r = \frac{D}{\int_0^\infty c(t) \, dt}.
\]

Since the total dye \( D \) is known (the amount injected), \( c(t) \) is measured by the test, and \( \int_0^\infty c(t) \, dt \) can be computed by Riemann sums,* we obtain flow rate \( r \).

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* Since \( c(t) = 0 \) after all the dye has passed, \( \int_0^\infty c(t) \, dt \) can be cut off to a finite integral.