Fundamental Theorem of Calculus

Integral as antiderivative. In §4.1, we were given a velocity function $v(t)$, and we wanted to determine the distance traveled over a given time interval $t \in [a,b]$. The answer was an integral defined as a Riemann sum, adding up (velocity)×(time) over many small time increments of length Δt:

$$\text{distance traveled} \;=\; s(b)-s(a) \;=\; \lim_{\Delta t\to 0}\sum_{i=1}^{n} v(t_i)\,\Delta t \;=\; \int_a^b v(t)\,dt.$$

Assuming initial position $s(a)=0$, and taking $b=x$, a variable endpoint, this means:*

$$s(x) \;=\; \int_a^x v(t)\,dt.$$

Since the rate of change of position is velocity, $s'(x) = v(x)$, this always computes an antiderivative function for $v(t)$, even if it is impossible to get an antiderivative algebraically by reversing differentiation formulas.

First Fundamental Theorem. Stating the above formally:

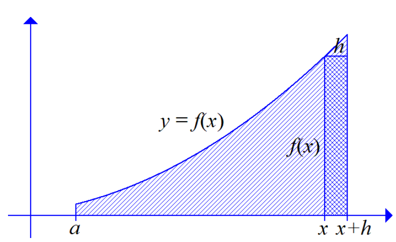
> *Theorem:* Let $f(x)$ be continuous for all $x \in [a,b]$ and define the function:†
>
> $$I(x) \;=\; \int_a^x f(t)\,dt.$$
>
> Then $I'(x) = f(x)$ for $x \in (a,b)$, and $I(x)$ is the unique antiderivative of $f(x)$ with $I(a) = 0$.

In more general physical terms: the rate of change of a cumulative effect up to some time is the strength of the effect at that time.

Proof. In a rigorous argument, we cannot use our physical intuition about velocity and position, and we do not even know if there exists any anti-derivative function. Rather, we define the candidate anti-derivative: $I(x) = \int_a^x f(x)\,dx$, and we compute its derivative from the definition: $I'(x) = \lim_{h\to 0}\frac{I(x+h)-I(x)}{h}$. We have:

$$\frac{I(x+h)-I(x)}{h} \;=\; \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right) \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt\,,$$

since $\int_a^{x+h} = \int_a^x + \int_x^{x+h}$ for all h (even $h<0$).



Notes by Peter Magyar `magyar@math.msu.edu`

*We use the new variable x to avoid $s(t) \overset{??}{=} \int_a^t v(t)\,dt$, which would imply nonsense like $s(2) \overset{??}{=} \int_a^2 v(2)\,d2$.

†Again, we must use different letters for the limit of integration x and the variable of integration t.

Geometrically, we see that if h is small enough, the region above $[x, x{+}h]$ is approximately a rectangle with height $f(x)$ and width h, so $\int_x^{x+h} f(x)\,dx \approx f(x)h$, and:

$$I'(x) \;\approx\; \frac{I(x{+}h) - I(x)}{h} \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\approx\; \tfrac{1}{h}(f(x)h) \;=\; f(x),$$

with approximations turning into equalities as $h \to 0$, as claimed by the Theorem.

However, geometric inspection is also insufficient for a proof, because any picture only shows a particular case, and is not numerically precise. To control errors, we take the absolute minimum value N and the absolute maximum value M of the continuous function $f(x)$ on $[x, x{+}h]$, using the Extremal Value Theorem (§3.1).[‡] (To indicate that these depend on h, we write N_h, M_h.) Now, $N_h \le f(t) \le M_h$ for $t \in [x, x{+}h]$, so by the Bounds Rule for integrals (§4.2) we have:

$$((x{+}h){-}x)N_h \;\le\; \int_x^{x+h} f(t)\,dt \;\le\; ((x{+}h){-}x)M_h \quad\Longrightarrow\quad N_h \;\le\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\le\; M_h.$$

As h gets very small, the interval $[x, x{+}h]$ gets closer and closer to the single point x, and the absolute minimum and maximum over this tiny interval must approach $f(x)$ by continuity: that is, $\lim_{h\to 0} N_h = \lim_{h\to 0} M_h = f(x)$. Also, by the above we have:

$$N_h \;\le\; \frac{I(x{+}h) - I(x)}{h} \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt \;\le M_h\,.$$

Applying the Squeeze Theorem for limits (§1.6), we find what we wanted:

$$I'(x) \;=\; \lim_{h\to 0} \frac{I(x{+}h) - I(x)}{h} \;=\; \lim_{h\to 0} N_h \;=\; \lim_{h\to 0} M_h \;=\; f(x)\,,$$

As for the uniqueness part of the conclusion, it is clear that $I(a) = \int_a^a f(t)\,dt = 0$, and there is a unique antiderivative with this initial value by the Antiderivative Theorem (§3.9), which is a version of the Uniqueness Theorem (§3.2). Note how we have used almost all of our previous theory in proving this culminating Theorem.

Derivative of integral functions. The above Theorem can be stated as a Basic Derivative formula for $I(x) = \int_a^x f(t)\,dt$, where $f(t)$ is continuous:

$$I'(x) \;=\; \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) \;=\; f(x)\,.$$

Here a is any constant, x is the input variable, and t is a "dummy variable" which only has meaning inside the integral.

For another function $g(x)$, we can take its composition with $I(x)$. Then the above Basic Derivative together with the Chain Rule (§2.5) implies:

$$I(g(x))' \;=\; \frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) \;=\; I'(g(x))\cdot g'(x) \;=\; f(g(x))\cdot g'(x)\,.$$

[‡]Here we assume $h > 0$. The case $h < 0$ is the same except for a few sign changes.

EXAMPLE: Find the derivative of $F(x) = \int_{2x}^{x^3} \sin(x)\,dx$. We have:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\int_{2x}^{x^3} \sin(x)\,dx\right) = \frac{d}{dx}\left(\int_{0}^{x^3} \sin(x)\,dx - \int_{0}^{2x} \sin(x)\,dx\right) \\ &= \sin(x^3)\cdot(x^3)' - \sin(2x)\cdot(2x)' = 3x^2\sin(x^3) - 2\sin(2x)\,. \end{aligned}$$

EXAMPLE: Find the derivative of $F(x) = \int_{2a}^{b^3} \sin(t)\,dt$. Here a, b are constants, and hence so are $2a, b^3$. In fact, the right hand side does not depend on the variable x, and is a constant function with derivative $F'(x) = 0$! This also follows from the Chain Rule, since $\sin(2a)\cdot 2(a)' = 0$ and $\sin(b^3)\cdot(b^3)' = \sin(b^3)\cdot 3b^2\cdot(b)' = 0$.

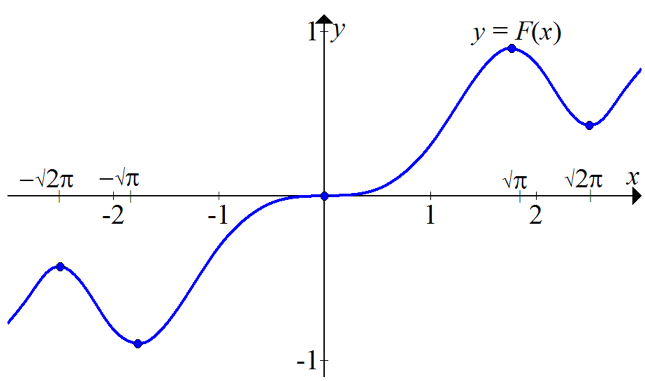
Sketching integral functions. Since an antiderivative $I(x) = \int_a^x f(t)\,dt$ might be a completely new function for which no elementary formula is possible, it might seem mysterious. However, we can find its values numerically with sufficient accuracy by computing Riemann sums on a spreadsheet, and plot these to get a good idea of the graph.

A geometric strategy is to use the derivative $I'(x) = f(x)$ for sketching $y = I(x)$, as in §3.3 and §3.5. That is, the *slope* of the graph $y = I(x)$ is given by the *height* of $y = f(x)$.

EXAMPLE: Graph the function $I(x) = \int_0^x \sin(t^2)\,dt$. The critical points of $I(x)$ are solutions of $I'(x) = 0$ or undefined, i.e. $f(x) = \sin(x^2) = 0$ (defined for all x). This happens when $x^2 = 2k\pi$ for any integer k, so the critical points are $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. Sign chart:

x		$-\sqrt{2\pi}$		$-\sqrt{\pi}$		0		$\sqrt{\pi}$		$\sqrt{2\pi}$	
$I'(x)$	$+$	0	$-$	0	$+$	0	$+$	0	$-$	0	$+$
$I(x)$	$\nearrow$	-0.43	$\searrow$	-0.89	$\nearrow$	0	$\nearrow$	0.89	$\searrow$	0.43	$\nearrow$

For inflection points, we solve $I''(x) = 0$, i.e. $f'(x) = 2x\cos(x^2) = 0$, so $x = 0, \pm\sqrt{\frac{\pi}{2}}, \pm\sqrt{\frac{3\pi}{2}}, \ldots$. Thus, the general shape of the graph is clear, and we can get specific points $(b, I(b))$ from computing a Riemann sum for $\int_0^b \sin(t^2)\,dt$.



From the 180° rotational symmetry of the graph, it looks like $I(x)$ is an odd function,

$I(-x) = -I(x)$. This is because $f(x) = \sin(x^2)$ is an even function, $f(-x) = f(x)$, so:

$$\begin{aligned} I(-b) &= \int_0^{-b} \sin(t^2)\,dt \;=\; -\int_{-b}^{0} \sin(t^2)\,dt \\ &= -(\text{area under } y = \sin(x^2) \text{ above } x \in [-b, 0]) \\ &= -(\text{area under } y = \sin(x^2) \text{ above } x \in [0, b]) \\ &= -\int_0^b \sin(t^2)\,dt \;=\; -I(b) \end{aligned}$$

Second Fundamental Theorem. This is a trick to easily evaluate many integrals, which we already used to find some exact values in §4.1.

> *Theorem:* Suppose $F(x)$ is some known antiderivative with $F'(x) = f(x)$. Then:
>
> $$\int_a^b f(x)\,dx \;=\; F(b) - F(a)\,.$$
>
> That is, if $f(x)$ is the rate of change of $F(x)$, then the integral $\int_a^b f(x)\,dx$ is the total change of $F(x)$ from $x = a$ to b.

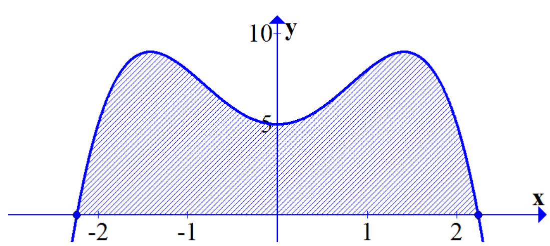
Put another way: the cumulative effect of a rate of change is a total change.

Proof. Since $F(x)$ is a particular antiderivative of $f(x)$, the Uniqueness Theorem (§3.9, §3.2) says that the general antiderivative is $F(x) + C$ for any constant C. But the First Fundamental Theorem says the integral function $I(x) = \int_0^x f(t)\,dt$ is also an antiderivative of $f(x)$, so we must have $I(x) = F(x) + C$. Since we know the initial condition $I(a) = \int_a^a f(t)\,dt = 0$, we get $I(a) = F(a) + C = 0$, and $C = -F(a)$. Therefore $I(x) = F(x) - F(a)$ and $\int_a^b f(x)\,dx = I(b) = F(b) - F(a)$ as desired.[§]

EXAMPLE: Evaluate the integral: $\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx$. Reversing our Derivative Rules as we did in §3.9, we see that $F(x) = 5x{+}\frac{4}{3}x^3{-}\frac{1}{5}x^5$ is an antiderivative. By the Theorem:

$$\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx \;=\; F(\sqrt{5}) - F(-\sqrt{5}) \;=\; \tfrac{20}{3}\sqrt{5} - (-\tfrac{20}{3}\sqrt{5}) \;=\; \tfrac{40}{3}\sqrt{5} \;\approx\; 29.81$$

EXAMPLE: Determine the area under the curve $y = 5{+}4x^2{-}x^4$ and above the x-axis.



We must find the limits of integration, which are the x-intercepts of the graph. Subtituting $u = x^2$, the equation becomes $5 + 4u - u^2 = 0$, which we can solve by the Quadratic Formula as $u = -1$ or 5, so $x = \pm\sqrt{u} = \pm\sqrt{5}$. Thus the area is $\int_{-\sqrt{5}}^{\sqrt{5}} 5{+}4x^2{-}x^4\,dx = \frac{40}{3}\sqrt{5}$. (Check: Graph's average height ≈ 7, base ≈ 4, so area ≈ 28, agreeing with above.)

[§] The variable of integration, x or t, is irrelevant, provided it doesn't conflict with the limits of integration.