Precise definition. We have defined the integral $\int_a^b f(x)\,dx$ as a number approximated by Riemann sums. The integral is useful because, given a velocity function, it computes distance traveled; given a graph, it computes an area between the graph and the x-axis. More generally, given a varying rate of change, the integral computes the total change; or given a varying linear influence, the integral computes its cumulative effect.

The parts of the notation $\int_a^b f(x)\,dx$ have their own names: $\int$ is the integral sign; a is the *lower limit of integration;*[*] b is the *upper limit of integration*; $f(x)$ is the *integrand*; and x is the *variable of integration*. Note that the variable of integration is named only for convenience, and changing it does not change the value of the integral: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$.

Now we give the formal definition of integrals on the numerical level, as we did for limits in §1.7 and derivatives in §2.1.

> *Definition:* Given a function $f(x)$ and numbers $a \le b$.
>
> - For each positive integer n, we divide the interval $x \in [a, b]$ into n increments of width $\Delta x = \frac{b-a}{n}$, with division points:
>
> $$ a \;<\; a+\Delta x \;<\; a+2\Delta x \;<\; \cdots \;<\; a+n\Delta x = b, $$
>
> and we choose sample points $x_1, \ldots, x_n$ with x_i anywhere in the ith increment: $a+(i-1)\Delta x \le x_i \le a+i\Delta x$. Then we let:
>
> $$ \int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \;=\; \lim_{n\to\infty} f(x_1)\Delta x + \cdots + f(x_n)\Delta x. $$
>
> - The function $f(x)$ is *integrable* over $[a, b]$ whenever the above limit exists for every possible choice of sample points x_i.[†]

Integrable functions. Most functions are integrable unless they have a vertical asymptote. To be precise:

> *Theorem:* Assume $f(x)$ is continuous for all $x \in [a, b]$, except possibly at a finite list of removable or jump discontinuities (see §1.8).
>
> Then $f(x)$ is integrable, meaning its Riemann sums converge to a well-defined limit $L = \int_a^b f(x)\,dx$ for any choice of sample points.

This is proved in courses on Real Analysis.

To understand integrability better, let's examine the non-integrable function $f(x) = \frac{1}{x^2}$ on $[a, b] = [0, 1]$. (We arbitrarily set $f(0) = 0$ to make $f(x)$ defined for

---

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[*] Here we use "limit" to mean a boundary, *not* a value approached by approximations.

[†] Even more formally, $\int_a^b f(x)\,dx = L$ means that for any error tolerance $\varepsilon > 0$, there is some lower bound N such that any Riemann sum with more than N terms is forced close to L within an error of ε: that is, $n > N$ forces $\left| \sum_{i=1}^n f(x_i)\Delta x - L \right| \;<\; \varepsilon$.

all $x \in [0, 1]$.) The function has a vertical asymptote discontinuity at $x = 0$, so the Theorem does not apply. If we attempt to compute $\int_0^1 \frac{1}{x^2}\, dx$ by a Riemann sum with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and sample points $x_i = a + i\Delta x = \frac{i}{n}$, we get:

$$\sum_{i=1}^{n} \tfrac{1}{x_i^2} \cdot \Delta x \;=\; \sum_{i=1}^{n} \tfrac{n^2}{i^2} \cdot \tfrac{1}{n} \;=\; \sum_{i=1}^{n} n \cdot \tfrac{1}{i^2} \;=\; n + \tfrac{n}{4} + \cdots \;>\; n\,.$$

Thus, as $n \to \infty$, the Riemann sum also gets larger and larger, and does not approach a finite limit. Geometrically, this means there is an infinite area under the curve and above the interval $[0, 1]$ on the x-axis.

Negative integrand. So far, we have considered $\int_a^b f(x)\, dx$ with positive integrand $f(x) \geq 0$, in which case the integral is a *positive* number (in fact, an area). Now suppose $f(x) \leq 0$: our definition of the integral still makes sense, but it gives a *negative* number. For example, for the constant function $f(x) = -1$, we have:

$$\begin{aligned}
\int_1^3 (-1)\, dx \;&=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} (-1)\left(\tfrac{3-1}{n}\right) \\
&=\; \lim_{n\to\infty} \left(\tfrac{-2}{n}\right) \sum_{i=1}^{n} 1 \;=\; \lim_{n\to\infty} \left(\tfrac{-2}{n}\right) \cdot n \;=\; -2\,.
\end{aligned}$$

Geometrically, we think of each term $f(x_i)\,\Delta x$ as (height)×(width) with a "negative height" $f(x_i) \leq 0$, and we count this as a "negative area". For a general graph $y = f(x)$ passing above and below the x-axis, $\int_a^b f(x)\, dx$ computes the "signed area" between the graph and the interval $[a, b]$ on the x-axis, with regions above the x-axis counting as positive area, and regions below counting negative.

EXAMPLE: We could evaluate the integral $\int_0^3 (2-x)\, dx$ with Riemann sums as above, but it is easier geometrically. The function $f(x) = 2-x$ has x-intercept $x = 2$; it is positive for $x \in [0, 2]$ and negative for $x \in [2, 3]$. Thus the integral is the area of the triangle above $[0, 2]$, *minus* the area of the triangle below $[2, 3]$:
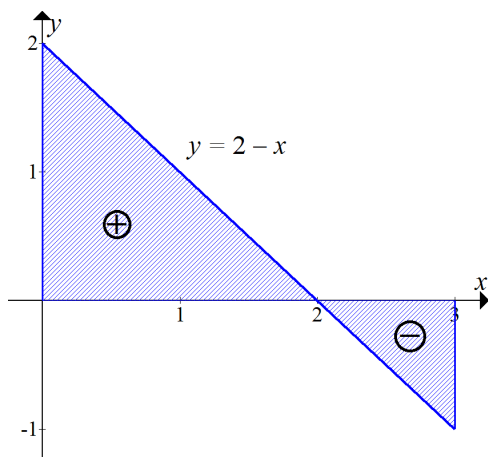
But (triangle area) $= \frac{1}{2}$(base)×(height), so $\int_0^3 (2-x)\, dx = \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1) = \frac{3}{2}$.

Reversing limits of integration. If we take the limits of integration to be the same, $a = b$, then $\Delta x = \frac{b-a}{n} = 0$, so every Riemann sum is zero, and we get:

$$\int_a^a f(x)\, dx \;=\; 0\,.$$

This is clear geometrically, since the area above a one-point interval $[a, a]$ is zero.

Next, we give a meaning to switching the two limits of integration, by defining:

$$\int_b^a f(x)\, dx \;=\; -\int_a^b f(x)\, dx\,.$$

Geometrically, in $\int_b^a f(x)\, dx$ with $a < b$, we imagine x running backward from b to a with a negative increment $\Delta x = \frac{a-b}{n} < 0$. Since each term $f(x_i)\Delta x$ has "negative width" Δx, the integral becomes the negative of $\int_a^b f(x)\, dx$. If $f(x_i)$ is also negative, then both width and height are negative and the integral is positive: for example, $\int_3^1 (-1)\, dx = -\int_1^3 (-1)\, dx = -(-2) = 2$.

We bother with this definition only so as to improve the Splitting Rule below.

Integral Rules. At the end of §4.1 Part 2 we gave a direct computation of an integral as a limit of Riemann sums. In later sections, we will find much better algebraic methods to compute integrals using the Fundamental Theorems of Calculus. For now, we will rely on a few Basic Integrals, and the following Rules to combine them. Let a, b, c be any points on the x-axis; $f(x), g(x)$ any integrable functions; and A, B, C any constants. Then we have:

- *Sum:* $\displaystyle\int_a^b f(x){+}g(x)\, dx \;=\; \int_a^b f(x)\, dx \;+\; \int_a^b g(x)\, dx\,.$

- *Difference:* $\displaystyle\int_a^b f(x){-}g(x)\, dx \;=\; \int_a^b f(x)\, dx \;-\; \int_a^b g(x)\, dx\,.$

- *Constant Multiple:* $\displaystyle\int_a^b C\, f(x)\, dx \;=\; C\cdot\int_a^b f(x)\, dx\,.$

- *Domination:* If $f(x) \le g(x)$, then: $\displaystyle\int_a^b f(x)\, dx \;\le\; \int_a^b g(x)\, dx\,.$

- *Bounds:* If $A \le f(x) \le B$, then: $\displaystyle (b{-}a)A \;\le\; \int_a^b f(x)\, dx \;\le\; (b{-}a)B\,.$

- *Splitting:* $\displaystyle\int_a^c f(x)\, dx \;=\; \int_a^b f(x)\, dx \;+\; \int_b^c f(x)\, dx\,.$

Proof. The Sum, Difference, and Constant Multiple Rules follow directly from the corresponding rules for summations in §4.1 Part 2, applied to Riemann sums.

The Bounds Rule makes sense geometrically because $0 \le A \le f(x) \le B$ means the graph $y = f(x)$ is above the line $y = A$ and below $y = B$. Thus the area $\int_a^b f(x)\, dx$ below $y = f(x)$ and above $[a, b]$ contains a rectangle with

(width)$\times$(height) $= (b-a)A$, and the area is contained inside a rectangle with (width)$\times$(height) $= (b-a)B$.
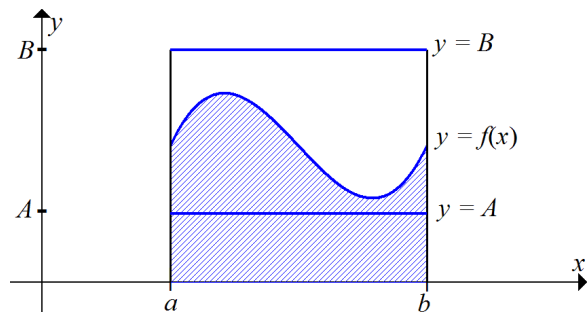
Computing formally, $A \le f(x_i) \le B$ implies $\sum_{i=1}^{n} A\,\Delta x \le \sum_{i=1}^{n} f(x_i)\Delta x \le \sum_{i=1}^{n} B\,\Delta x$. Hence:

$$\sum_{i=1}^{n} A\,\Delta x \;=\; \sum_{i=1}^{n} A \cdot \tfrac{b-a}{n} \;=\; \tfrac{(b-a)A}{n}\sum_{i=1}^{n} 1 \;=\; \tfrac{(b-a)A}{n}\cdot n \;=\; (b-a)A,$$

and similarly for the upper bound. Taking limits as $n \to \infty$ gives the desired inequalities. The Domination Rule is similar.

The Splitting Rule is intuitive when $a \le b \le c$. The interval $[a,c]$ splits as the union of two sub-intervals, $[a,b] \cup [b,c]$, so the area above $[a,c]$ is the sum of the areas above $[a,b]$ and $[b,c]$, i.e. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

Furthermore, because of our extended definition of integrals, the Splitting Rule is valid no matter what the relative positions of a, b, c. For example, if $a \le c \le b$, then $[a,b] = [a,c]\cup[c,b]$ and clearly $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$. Moving $\int_c^b$ to the other side, we get:

$$\int_a^c f(x)\,dx \;=\; \int_a^b f(x)\,dx - \int_c^b f(x)\,dx \;=\; \int_a^b f(x)\,dx + \int_b^c f(x)\,dx\,,$$

so the very same Splitting Rule applies: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

Another example: if $a = c$, the Splitting Rule says:

$$\int_a^a f(x)\,dx \;=\; \int_a^b f(x)\,dx + \int_b^a f(x)\,dx,$$

which is true since both sides are zero.

Basic Integrals:

$$\int_a^b 1\,dx \;=\; b - a, \qquad \int_a^b x\,dx \;=\; \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2, \qquad \int_a^b x^2\,dx \;=\; \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3.$$

Later, we will easily evaluate these integrals by the Fundamental Theorems. For now, we can prove them directly from the Basic Summations in §4.1 Part 2. For

the third and hardest formula, we take increment $\Delta x = \frac{b-a}{n}$, sample points $x_i = a + i\Delta x$, and $f(x_i) = (a+i\Delta x)^2 = a^2 + 2ai\,\Delta x + i^2(\Delta x)^2$, giving Riemann sum:

$$
\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} a^2 \Delta x + 2ai(\Delta x)^2 + i^2(\Delta x)^3 \\
&= a^2 \Delta x \sum_{i=1}^{n} 1 + 2a(\Delta x)^2 \sum_{i=1}^{n} i + (\Delta x)^3 \sum_{i=1}^{n} i^2 \\
&= a^2 \frac{(b-a)}{n} n + 2a \frac{(b-a)^2}{n^2} \tfrac{1}{2}n(n+1) + \frac{(b-a)^3}{n^3} \tfrac{1}{6}n(n+1)(2n+1) \\
&= a^2(b-a) + 2a(b-a)^2(\tfrac{1}{2}+\tfrac{1}{2n}) + (b-a)^3(\tfrac{1}{3}+\tfrac{1}{2n}+\tfrac{1}{6n^2})
\end{aligned}
$$

Taking the $\lim_{n\to\infty}$, the terms with n in the denominator disappear, and we get:

$$
\int_a^b x^2\,dx \;=\; a^2(b-a) + 2a(b-a)^2(\tfrac{1}{2}) + (b-a)^3(\tfrac{1}{3}) \;=\; \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3.
$$

Examples.

- Evaluate the integral:

$$
\int_{-2}^{7} (3t-5)^2\,dt.
$$

We use the Integral Rules to reduce the problem to Basic Integrals. Since $\int f(x)^2\,dx$ is NOT equal to $(\int f(x)\,dx)^2$, we must expand the integrand and apply the Sum, Difference, and Constant Multiple Rules:

$$
\begin{aligned}
\int_{-2}^{7}(3t-5)^2\,dt &= \int_{-2}^{7}(9t^2 - 30t + 25)\,dt \\
&= 9\int_{-2}^{7} t^2\,dt \;-\; 30\int_{-2}^{7} t\,dt \;+\; 25\int_{-2}^{7} 1\,dt \\
&= 9\left(\tfrac{1}{3}7^3 - \tfrac{1}{2}(-2)^3\right) - 30\left(\tfrac{1}{2}7^2 - \tfrac{1}{2}(-2)^2\right) + 25\left(7-(-2)\right) \\
&= 603\,.
\end{aligned}
$$

Note that the variable of integration (t or x) is irrelevant.

- Find an upper bound for:

$$
\int_{-2}^{7} (3x-5)^2\,(1+\cos(x^2))\,dx.
$$

That is, we do not ask for an exact value, only an overestimate. We know that $1 + \cos(x^2) \le 2$, so the Domination Rule gives:

$$
\int_{-2}^{7}(3x-5)^2(1+\cos(x^2))\,dx \;\le\; \int_{-2}^{7}(3x-5)^2(2)\,dx \;=\; 2(603) \;=\; 1206\,.
$$