Review. The derivative of $y = f(x)$ has four levels of meaning:

- Physical: If y is a quantity depending on x, the derivative $\frac{dy}{dx}|_{x=a}$ is the rate of change of y per tiny change in x away from a.
- Geometric: $f'(a)$ is the slope of the graph $y = f(x)$ near the point $(a, f(a))$, or the slope of the tangent line at that point, $y = f(a) + f'(a)(x-a)$.
- Numerical: Approximate by the difference quotient, $f'(a) \approx \frac{\Delta f}{\Delta x} = \frac{f(x)-f(a)}{x-a}$ for x near a. This gives the linear approximation $f(x) \approx f(a)+f'(a)(x-a)$.
- Algebraic: Defining $f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, we prove Basic Derivatives and Derivative Rules to find $f'(x)$ for any formula $f(x)$.

Problems usually originate on the physical or geometric levels, then we translate them to the numerical or algebraic levels to solve them. For example, to find the hill tops of a given curve $y = f(x)$, a geometric problem, we consider that they must have horizontal tangents, so we take the derivative $f'(x)$ and solve for the critical points $f'(x) = 0$ algebraically, or numerically with Newton's Method.

In the previous chapter §3.9, we introduced the reverse of the derivative, the *antiderivative*. In this chapter, we will see that it has all the above levels of meaning, and connecting them will allow us to solve many new problems.

Distance problem. In §3.9, we reversed the *algebraic* derivative operation: that is, we could often recognize a given a function $f(x)$ as the derivative of some familiar function $F(x)$, obtaining an algebraic antiderivative. But this does not always work: there are many functions which are not the derivative of any formula we know, for example $f(x) = \frac{1}{x}$ or $\sqrt{x^3+1}$ or $\sin(x^2)$.*

Let us consider this on the *physical* level: if we take $f(t) = v(t)$ to be a velocity function, then the antiderivative should be the corresponding position function $F(t) = s(t)$, since velocity is the rate of change of position, $s'(t) = v(t)$. Imagine a toy car on a track which starts out at time $t = 0$ at the starting line $s(0) = 0$, and adjusts its velocity according to $v(t)$. Even if we have no algebraic formula for $s(t)$, nevertheless the car does have a position, so there must exist an antiderivative, a new function we have not imagined before.

To compute this new position function $s(t)$, we work *numerically*, taking a limit of approximations as we did in computing the derivative $\frac{ds}{dt} = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t}$. To illustrate, we take the simple velocity function $v(t) = t^2$, and compute $s(2)$, the distance traveled from $t = 0$ to $t = 2$:

$$s'(t) \;=\; v(t) \;=\; t^2, \qquad s(0) = 0, \qquad s(2) \;= ??$$

Since distance = velocity × time, we can say very roughly:

$$s(2) \;\approx\; v(2)\,\Delta t \;=\; 2^2\,(2-0) \;=\; 8.$$

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* We will eventually learn that $\frac{1}{x}$ is the derivative of the logarithm $\log(x)$, but the others really are not the derivative of any formula.

This would be exact if the velocity held constant at $v(2) = 4$ for the whole time, but in fact the car starts from a standstill $v(0) = 0$, so this is a gross overestimate.

For a good approximation, we split the time interval $[0, 2]$ into $n = 20$ increments[†] of size $\Delta t = 0.1$, with dividing points:

$$0.0 < 0.1 < 0.2 < \cdots < 1.8 < 1.9 < 2.0.$$

We approximate a distance increment during each time increment, and add these up to get the total distance traveled; this is called a *Riemann sum*:

$$\begin{aligned} s(2) &\approx v(0.1)\Delta t + v(0.2)\Delta t + \cdots + v(1.9)\,\Delta t + v(2.0)\,\Delta t \\ &= (0.1)^2\,(0.1) + (0.2)^2\,(0.1) + \cdots + (1.9)^2\,(0.1) + (2.0)^2\,(0.1) \\ &\approx 2.9\,. \end{aligned}$$

Here we sample the velocity $v(t)$ at the end of each increment: for example, the first sample point is $t = 0.1$, the right endpoint of $t \in [0.0, 0.1]$. This is still an *overestimate* (upper Riemann sum), since the velocity is slightly less at the beginning of each increment than at the end.

To get an *underestimate* (lower Riemann sum), we should sample velocity at the beginning of each increment, where it is smallest:

$$\begin{aligned} s(2) &\approx v(0.0)\Delta t + v(0.1)\Delta t + \cdots + v(1.8)\,\Delta t + v(1.9)\,\Delta t \\ &= (0.0)^2\,(0.1) + (0.1)^2\,(0.1) + \cdots + (1.8)^2\,(0.1) + (1.9)^2\,(0.1) \\ &\approx 2.5 \end{aligned}$$

As we take more and more increments of smaller and smaller size, all estimates converge on a limiting value, which is the exact position $s(2)$.

For this simple function $v(t) = t^2$, we can compare the numerical answers with our known algebraic solution: $s(t) = \frac{1}{3}t^3$ is the unique antiderivative with $s(0) = 0$, and we have:

$$s(2) \;=\; \tfrac{1}{3}(2^3) \;=\; \tfrac{8}{3} \;\approx\; 2.66.$$

which is indeed between the lower and upper estimates above. In fact, the average of the two estimates is $\frac{2.9+2.5}{2} = 2.7$, which is the correct answer rounded to 1 decimal place.

The integral. Applied generally to any velocity $v(t)$ over any interval $t \in [0, b]$, this method specifies the value of the position $s(b)$ as a limit. We introduce a new notation for this limit, the *integral* of $v(t)$ from $t = 0$ to b:

$$s(b) \;=\; \int_0^b v(t)\,dt \;=\; \lim_{\Delta t \to 0} v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t.$$

The integral symbol $\int$ is an elongated S standing for the *sum* of n terms; $v(t)$ stands for all the sample values $v(t_1), \ldots, v(t_n)$; and dt suggests a very small Δt for larger and larger $n \to \infty$.

[†] *Increment*: a small increase, a part added.

Sometimes $s(t)$ turns out to equal a known formula, sometimes it can only be computed approximately to any desired accuracy. In our example, we computed $s(2) = \int_0^2 t^2\,dt = \frac{8}{3} \approx 2.66$.

Generalizing further, suppose we are given any function $f(x)$ which we consider as the *rate of change* of an unknown function $F(x)$ for $x \in [a,b]$. Then we may compute the *total change* $F(b) - F(a)$ by the above method: that is, we compute the integral of $f(x)$ from $x = a$ to b:

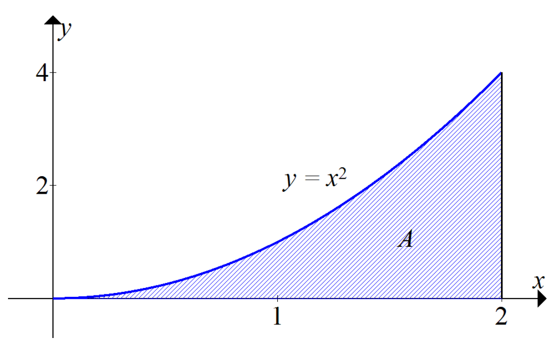
$$F(b) - F(a) \;=\; \int_a^b f(x)\,dx \;=\; \lim_{\Delta x \to 0} f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Here we split the interval $[a,b]$ into n increments of size $\Delta x = \frac{b-a}{n}$, and choose a sample point in each increment: $x_1, x_2, \ldots, x_n$ can be the left or right endpoints, or anywhere between. Each term approximates the incremental change in $F(x)$ as the rate of change $f(x_i)$ times the length of the increment Δx. Finally, we take the limit as $n \to \infty$ and $\Delta x \to 0$.

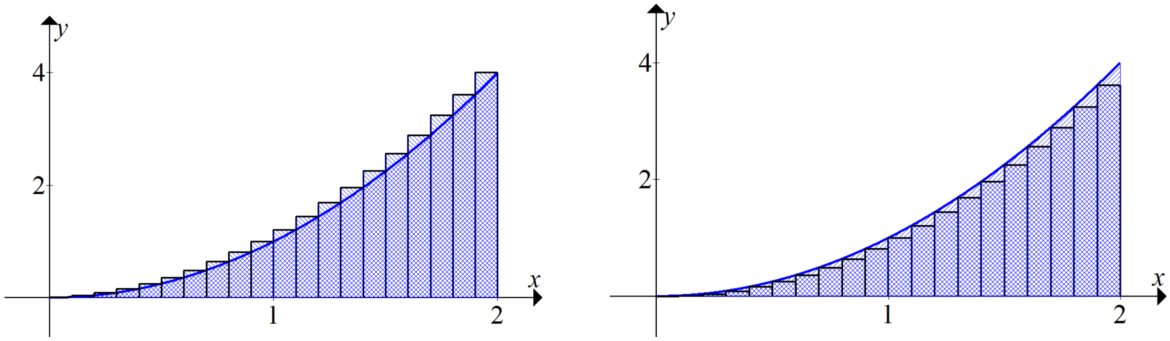
Cumulative effect of a varying influence. Another perspective: consider $f(x)$ as a varying *linear influence* on a variable $z = F(x)$, meaning that a small input increment Δx produces an output increment $\Delta z \approx f(x)\,\Delta x$. (This is the same as $f(x) \approx \frac{\Delta z}{\Delta x} \approx \frac{dz}{dx}$, i.e. $f(x)$ is the rate of change of z.) Examples: velocity influences position, acceleration influences velocity, pointwise density influences total mass, heating influences temperature, power consumption influences total energy use.

Then $\int_a^b f(x)\,dx$ computes the *cumulative effect* of this influence of $f(x)$ on z for $x \in [a,b]$, since it adds up all the Δz-increments $f(x_1)\Delta x + \cdots + f(x_n)\Delta x$.

Area problem. Now we come to one of the most surprising results in mathematics: the *geometric* interpretation of the integral. Suppose we have a function with $f(x) \geq 0$ for $x \in [a,b]$, and we wish to determine the area under the graph $y = f(x)$ and above the interval $[a,b]$ on the x-axis. For example, let us again take $f(x) = x^2$ over the interval $[0,2]$.



To approximate the area A, we cover it by 20 thin rectangles of width $\Delta x = 0.1$ (below at left):



The dividing points are again $0.0 < 0.1 < \cdots < 1.9 < 2.0$, and each rectangle reaches up to the graph at the right endpoint of an increment, giving heights $f(0.1), f(0.2), \ldots, f(2.0)$. The area A under the curve is close to the total area of the rectangles; adding up (height)$\times$(width) for each rectangle gives:

$$A \;\approx\; f(0.1)\,\Delta x + f(0.2)\,\Delta x + \cdots + f(2.0)\,\Delta x \;\approx\; 2.9\,.$$

This is an overestimate, since the rectangles slightly overshoot the curve. To get an underestimate, we take heights at the left endpoint of each increment, fitting the rectangles under the graph (above at right):

$$A \;\approx\; f(0.0)\,\Delta x + f(0.1)\,\Delta x + \cdots + f(1.9)\,\Delta x \;\approx\; 2.5\,.$$

Clearly, this is the same computation as we did before, so it has the same answer. That is, taking the limit of thinner and thinner rectangles gives:

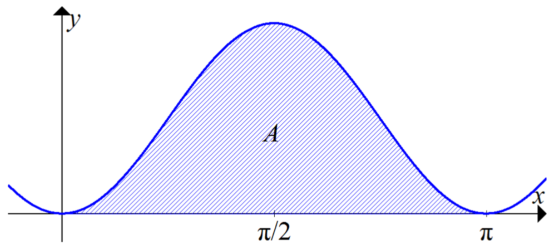
$$A \;=\; \int_0^2 x^2\,dx.$$

That is, the area under $y = x^2$ for $x \in [0,2]$ is the *same* as the distance traveled with velocity $v(t) = t^2$ during $t \in [0,2]$. Are you not amazed?

Why is this? Let us fix a, take $b = x$ to be a variable, and consider the area above the variable interval $[a,x]$ as a function $A(x)$. Then the rate of change of the area function is the height of the graph at the right endpoint: $A'(x) = f(x)$, since the greater the height, the taller the rightmost incremental rectangle, and the faster $A(x)$ increases. Thus, we can consider the height as a rate of change, and the area as a total change, which is just what the integral computes. (Also, we can consider the area as the cumulative effect of the graph height.)

We have seen the distance problem before in §2.7, when we used speedometer data to reconstruct odometer data, using the graph of velocity $v(t)$ to draw the graph of distance $s(t)$. We can now compute $s(t)$ more directly at a particular $t = x$ as the area under the $v(t)$ graph above the interval $t \in [0,x]$. (For consistency, we must look at the ft/sec scale for $v(t)$, not mph, since t is in sec.) This is how the $s(t)$ graphs were computed.

Approximating an integral. We compute the area A under one arch of $f(x) = \sin^2(x)$ to an accuracy of one decimal place:



We will compute:

$$A \;=\; \int_0^\pi \sin^2(x)\,dx \;\approx\; \sin^2(x_1)\,\Delta x + \sin^2(x_2)\,\Delta x + \cdots + \sin^2(x_n)\,\Delta x$$

for suitably large n, the corresponding small increment $\Delta x = \frac{b-a}{n} = \frac{\pi}{n}$, and appropriate sample points $x_1, \ldots, x_n$.

To make sure of the required accuracy, we will compute an overestimate and an underestimate (upper and lower Riemann sums). For an *overestimate*, we take $x_1, \ldots, x_n$ so that $\sin^2(x)$ is *largest* within each increment. These are not always the right endpoints, because the function is decreasing on the second half of the interval. Rather, for the increments within $[0, \frac{\pi}{2}]$, we take the right endpoints, and for the increments within $[\frac{\pi}{2}, \pi]$ we take the left endpoints. To get an *underestimate*, we take sample points where $\sin^2(x)$ is *smallest* within each increment, reversing the previous choices.

With a spreadsheet or computer algebra software, it is not difficult to take $n = 100$, $\Delta x = 0.01\pi$. The upper estimate is:

$$\begin{aligned}&\sin^2(0.01\pi)\,(0.01\pi) + \sin^2(0.02\pi)\,(0.01\pi) + \cdots + \sin^2(0.50\pi)\,(0.01\pi)\\ &\qquad + \sin^2(0.50\pi)\,(0.01\pi) + \sin^2(0.51\pi)\,(0.01\pi) + \cdots + \sin^2(0.99\pi)\,(0.01\pi) \;\approx\; 1.60\,.\end{aligned}$$

The lower estimate is:

$$\begin{aligned}&\sin^2(0.00\pi)\,(0.01\pi) + \sin^2(0.01\pi)\,(0.01\pi) + \cdots + \sin^2(0.49\pi)\,(0.01\pi)\\ &\qquad + \sin^2(0.51\pi)\,(0.01\pi) + \sin^2(0.52\pi)\,(0.01\pi) + \cdots + \sin^2(1.00\pi)\,(0.01\pi) \;\approx\; 1.54\,.\end{aligned}$$

Thus we get upper and lower bounds for our area A, and taking their average gives our best estimate:

$$1.54 < A < 1.60 \quad\Longrightarrow\quad A \;=\; 1.57 \pm 0.03\,.$$

As we have seen, the integral $F(b) = \int_a^b f(x)\,dx$ defines an antiderivative function numerically, whether or not we can find an algebraic antiderivative. But in §3.9 we were (just barely) able to find an algebraic antiderivative: $F(x) = \frac{1}{2}x - \frac{1}{4}\sin(2x)$, satisfying $F(0) = 0$. Since there can be only one such antiderivative, we find that:

$$\int_0^\pi \sin^2(x)\,dx \;=\; F(\pi) \;=\; \frac{\pi}{2} \;\approx\; 1.571\,,$$

so our numerical approximation is actually accurate to 2 decimal places.