Math 132 # Antiderivatives **Stewart §3.9**

Reversing differentiation. In many problems, especially physical ones, we are interested in some function $F(x)$, but we only know its derivative $F'(x) = f(x)$. We need to reverse the differentiation process to find the original $F(x)$, the *antiderivative* of $f(x)$, also called the *primitive* of $f(x)$.

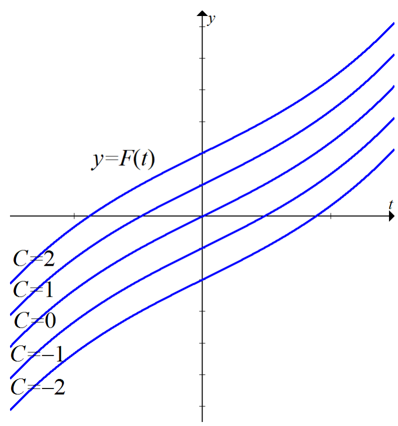
For example, suppose $F(t)$ represents the height of an object at time t, but we only know the velocity:

$$F'(t) = f(t) = t^2 + 2.$$

What was the original $F(t)$? Recalling our Basic Derivative $(t^n)' = nt^{n-1}$, we realize that $(\frac{1}{3}t^3)' = \frac{1}{3}(3t^2) = t^2$, so $F(t) = \frac{1}{3}t^3 + 2t$ works. But this is not the only possible answer because any constant term C disappears in the derivative, so the general answer is:

$$F(t) \ = \ \tfrac{1}{3}t^3 + 2t + C.$$

This family of functions is called the *general antiderivative*:



The non-uniqueness of $F(t)$ means that the velocity alone does not determine the height. But if we know the height at just one time, for example the initial height $F(0) = 5$, then we can adjust the constant C in a unique way to satisfy this requirement:

$$F(0) \ = \ \tfrac{1}{3}(0^3) + 2(0) + C \ = \ 5 \quad \Longrightarrow \quad C = 5.$$

That is, $F(t) = \frac{1}{3}t^3 + 2t + 5$ is the unique function with $F'(t) = t^2 + 2$ and $F(0) = 5$. We have solved an *initial value problem.* (See the Ballistic Equation, end of §2.7.)

Generalizing we have:

> *Definition.* $F(x)$ is an *antiderivative* of $f(x)$ means $F'(x) = f(x)$.
>
> *Antiderivative Theorem.* Assume $F(x)$ is some particular antiderivative of $f(x)$ for $x \in (a, b)$. Then:
>
> (a) The general antiderivative of $f(x)$ is $F(x) + C$ for any constant C. There are no other antiderivatives of $f(x)$.
>
> (b) For any $c \in (a, b)$ and any A, there is a unique antiderivative $\tilde{F}(x)$ satisfying the intial value problem $\tilde{F}'(x) = f(x)$ and $\tilde{F}(c) = A$.

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Proof. (a) This just rephrases §3.2 Uniqueness Theorem (b). That is, if $\tilde{F}(x)$ is any antiderivative, i.e. any function with $\tilde{F}'(x) = F'(x) = f(x)$, then the Uniqueness Theorem guarantees $\tilde{F}(x) = F(x) + C$ for some constant C.

(b) We are given a specific antiderivative $F(x)$ by hypothesis, so by part (a), a general antiderivative is $\tilde{F}(x) = F(x) + C$. If we require $\tilde{F}(c) = F(c) + C = A$, this determines C uniquely as $C = A - F(c)$, so we can only have $\tilde{F}(x) = F(x) + A - F(c)$. (See also §3.2 Uniqueness Theorem (c).) Q.E.D.

Antidifferentiation. This means the process of finding antiderivatives by reversing the rules for derivatives from §2.3–2.4. For every Basic Derivative of the form $F'(x) = f(x)$, we have a reverse Basic Antiderivative:

$F(x)$	$F'(x) = f(x)$
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\tan(x)\sec(x)$

$\Longrightarrow$

$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1} + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\tan(x)\sec(x)$	$\sec(x) + C$

Each general antiderivative has an arbitrary constant term C. Notice we do *not* know an antiderivative for $f(x) = \frac{1}{x} = x^{-1}$, since the formula $\frac{1}{-1+1}x^0$ does not make sense.

We can also reverse the Derivative Rules. Since the derivative of a sum is the sum of derivatives, the same is true for antiderivatives, and similarly for differences and constant multiples:

$$\begin{aligned} f(x) &= 7x^3 - x\sqrt{x} + \frac{3}{x^2} - \frac{4}{\cos^2(x)} \\ &= 7x^3 - x^{3/2} + 3x^{-2} - 4\sec^2(x), \end{aligned}$$

$$\begin{aligned} F(x) &= 7(\tfrac{1}{4}x^4) - \tfrac{1}{5/2}x^{5/2} + 3(\tfrac{1}{-1}x^{-1}) - 4\tan(x) + C \\ &= \tfrac{7}{4}x^4 - \tfrac{2}{5}x^2\sqrt{x} - \frac{3}{x} - 4\tan(x) + C. \end{aligned}$$

To verify this, just differentiate $F(x)$ to recover $f(x)$.

We can also reverse the Chain Rule: we know $(\sin(3x))' = \cos(3x) \cdot (3x)' = 3\cos(3x)$, so what $F(x)$ will have $F'(x) = \cos(3x)$?

$$f(x) = \cos(3x) \qquad \Longrightarrow \qquad F(x) = \tfrac{1}{3}\sin(3x) + C.$$

On the other hand, the derivative of a product is NOT the product of derivatives (§2.3), so the antiderivative of a product is NOT the product of antiderivatives. (Similarly for quotients.) We will learn how to handle these later, but for now, we can sometimes antidifferentiate products or quotients if we can expand them into sums of Basic Antiderivatives.

$$\begin{aligned} f(x) &= \frac{x+4}{\sqrt{x}} = \frac{x}{\sqrt{x}} + \frac{4}{\sqrt{x}} = x^{1/2} + 4x^{-1/2} \\ F(x) &= \tfrac{1}{3/2}x^{3/2} + 4(\tfrac{1}{1/2}x^{1/2}) + C = \tfrac{2}{3}x\sqrt{x} + 8\sqrt{x} + C. \end{aligned}$$

EXAMPLE. Find the antiderivative of $f(x) = \sin^2(x)$. This is a product of $\sin(x)$ with itself, and we need to expand it somehow in terms of Basic Antiderivatives. A clever idea: in the identity

$$\cos(2x) \;=\; \cos^2(x) - \sin^2(x) \;=\; (1 - \sin^2(x)) - \sin^2(x) \;=\; 1 - 2\sin^2(x),$$

we can solve for $\sin^2(x)$, so that:

$$\begin{aligned} f(x) &= \sin^2(x) \;=\; \tfrac{1}{2} - \tfrac{1}{2}\cos(2x), \\ F(x) &= \tfrac{1}{2}x - \tfrac{1}{2}(\tfrac{1}{2}\sin(2x)) + C \;=\; \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + C. \end{aligned}$$

Second derivative initial value problem. Suppose a launching rocket reaches height $s(t)$ at time t. Suppose we are given the acceleration function $a(t) = t + 1$, and also height and velocity at time $t = 1$, namely $s(1) = 10$ and $v(1) = 3$. We wish to find the height and velocity functions $s(t)$ and $v(t)$.

Rephrasing, we know velocity is the rate of change of position (i.e. height), $v(t) = s'(t)$, and acceleration is the rate of change of velocity, $a(t) = v'(t) = s''(t)$. Thus, we must solve the initial value problem:

$$s''(t) = t + 1, \quad s(1) = 10, \quad s'(1) = 3.$$

First, we antidifferentiate $a(t) = t + 1$ to get $v(t) = \frac{1}{2}t^2 + t + C$. Thus, we have:

$$v(1) = \tfrac{1}{2}(1^2) + 1 + C = 3,$$

which we can solve to get $C = \frac{3}{2}$, so that:

$$v(t) = s'(t) = \tfrac{1}{2}t^2 + t + \tfrac{3}{2}.$$

Next we antidifferentiate $v(t)$ to get:

$$s(t) \;=\; \tfrac{1}{2}(\tfrac{1}{3}t^3) + \tfrac{1}{2}t^2 + \tfrac{3}{2}t + B \;=\; \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2 + \tfrac{3}{2}t + B,$$

where B is another constant (different from the previous C). Again, we can solve:

$$s(1) = \tfrac{1}{6}(1^3) + \tfrac{1}{2}(1^2) + \tfrac{3}{2}(1) + B = 10 \qquad \Longrightarrow \qquad B \;=\; \tfrac{47}{6}.$$

The final answer is:

$$s(t) \;=\; \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2 + \tfrac{3}{2}t + \tfrac{47}{6}.$$

This is the unique solution, with no arbitrary constants.