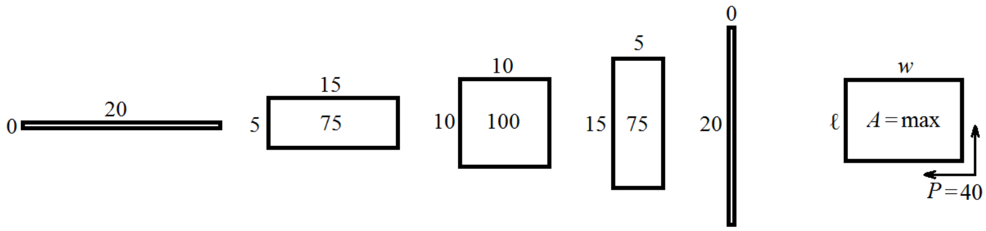
Optimization

Rectangle example. Suppose we have 40 meters of fence to make a rectangular corral. What length and width will fence off the largest area? The range of possiblilities is illustrated below:



It appears that the square with length and width $\ell = w = 10$ gives the maximum area $A = \ell w = 100$ m^2. To prove this algebraically, we note that the perimeter is constant, $P = 2\ell + 2w = 40$; so the length controls the width and also the area:

$$w = \tfrac{1}{2}(40-2\ell) = 20 - \ell, \qquad\qquad A = \ell w = \ell(20-\ell) = 20\ell - \ell^2\,.$$

That is, the quantity we aim to maximize, A, is a function of the variable ℓ, which is allowed to vary between $\ell = 0$ and $\ell = 20$ (corresponding to $w = 0$). This is a familiar problem: find the absolute maximum of

$$A(\ell) = 20\ell - \ell^2 \quad \text{over the interval } \ell \in [0, 20].$$

The critical points are given by $\frac{dA}{d\ell} = 20 - 2\ell = 0$, i.e. $\ell = 10$ with output $A(10) = 100$, and the endpoint outputs are $A(0) = 0$, $A(20) = 0$. The largest of these is the absolute maximum: $\ell = 10$ with $A(\ell) = 100$; also $w = 20 - \ell = 10$.

Method for optimization. We aim to find the maximum or minimum possible value of a target quantity within the constraints of a (usually geometric) situation.

1. Draw a picture labeled with numerical constant values and with letters for varying quantities, including: *controlling* variables to determine the shape; *constrained* variables required to have a fixed value; the *target* variable we aim to maximize or minimize.

2. Write equations relating variables according to the geometry of the picture.

3. Choose one of the controlling variables (say, x) as the *independent* variable, and write all other variables as functions of it by solving the above equations. Also determine the relevant domain $x \in [a, b]$, which is usually restricted by requiring all lengths to be positive.

4. Find the absolute maximum/minimum of the target variable over its domain, say $T = T(x)$ over $x \in [a, b]$. That is, solve $T'(x) = 0$ or undef, to find the critical points $x = c_1, c_2, \ldots$, as well as the endpoints $x = a, b$. Take the output values $T(x)$ at these candidate points: the largest/smallest output is the desired maximum/minimum.

5. If needed, find values of the other variables at the optimum x. Make sure the answer is physically plausible to check for mistakes.

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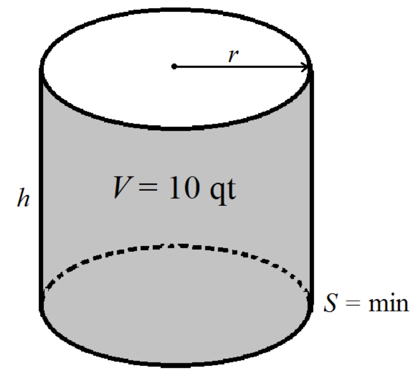
Bucket example. Consider a 10-quart bucket with cylindrical sides and circular bottom. What radius and height will minimize surface area of sides and bottom?

1. 

   The *target* variable is the surface area S (square inches), to be minimized. The *controlling* variables are radius r (inches) and height h (inches). The constant volume is expressed by the *constrained* variable $V = 10$ quarts; to make this comparable to the other variables, we must convert to $V = 577.5$ cubic inches.

2. Equations. The volume V is the base area πr^2 times the height h. For the surface S: the sides, if unrolled, form a rectangle with the same height h as the cylinder, and width equal to the perimeter of the bottom, $2\pi r$; and we also add the bottom area πr^2. Thus:

$$V \;=\; \pi r^2 h = 577.5\,, \qquad S \;=\; \pi r^2 + 2\pi r h \;=\; \text{min}.$$

3. Do we choose r or h as the *indendent* variable? Here r is harder to solve for, so we make it independent and solve for the other variables instead:

$$h \;=\; \frac{577.5}{\pi r^2}, \qquad S \;=\; \pi r^2 + 2\pi r\,\frac{577.5}{\pi r^2} \;=\; \pi r^2 + \frac{1155}{r}.$$

   The only restriction on r is $r > 0$. (Radius can be huge if height is correspondingly tiny: this is clearly not optimal, but still possible.) Thus, the domain is the open interval $r \in (0, \infty)$.

4. We must find the absolute minimum of $S(r) = \pi r^2 + \frac{1155}{r}$ over $r \in (0, \infty)$. To find the critical points:

$$\frac{dS}{dr} \;=\; 2\pi r - \frac{1155}{r^2} \;=\; 0 \quad\Longrightarrow\quad 2\pi r = \frac{1155}{r^2} \quad\Longrightarrow\quad r = \sqrt[3]{\frac{1155}{2\pi}} \;\approx 5.68\,.$$

   This is the only critical point, with output value $S(r) \approx 304\,.$

   Since the endpoint values $S(0)$ and $S(\infty)$ are not defined, we must consider the limiting values near these points: $\lim_{r\to 0^+} S(r) = \infty$ and $\lim_{r\to\infty} S(r) = \infty$. This means $S(r)$ has no absolute maximum, but can get as large as desired if we make r large or small enough.

   The remaining candidate must be the absolute minimum point, $r = \sqrt[3]{\frac{1155}{2\pi}}$.

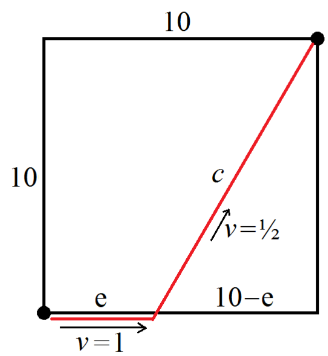
5. At the minimum point, the other variable is:

$$h \;=\; \frac{577.5}{\pi\left(\sqrt[3]{\frac{1155}{2\pi}}\right)^2} \;\approx\; 5.68\,.$$

In fact, we can simplify to show $h = r$, meaning the optimal bucket is twice as wide as it is high.

This is plausible. However, an actual 10-quart bucket has dimensions about as wide as it is high, about $r = \frac{1}{2}h = 4.5$ in, which uses more plastic than necessary. Try to explain what other factors might influence the design.

Ants example. A line of ants marches across a 10cm × 10cm square of carpet from the lower left to the upper right corner (where someone dropped a jellybean). Part of their path is along the edge next to the carpet, where their speed is 1 cm/sec, and part diagonally across the carpet, where their speed is $\frac{1}{2}$ cm/sec. What path should they take along the edge before entering the carpet, so as to minimize (a) the total distance; and (b) the total travel time.

1. 

   Controlling variables are e, the distance traveled along the edge, and c, the distance traveled across the carpet. The target variable to minimize for each question is: (a) total distance L in cm; and (b) total time T in sec.

2. Equations:

$$c^2 = 10^2 + (10{-}e)^2\,, \qquad L = e + c\,.$$

   Also, we know speed × time = distance, so time = distance/speed. The travel time along the edge is $e/1 = e$, along the carpet $c/\frac{1}{2} = 2c$, with total:

$$T \;=\; e + 2c\,.$$

3. The obvious independent variable is e, since we can easily write the other variables in terms of it, including the target variables:

$$c \;=\; \sqrt{10^2{+}(10{-}e)^2} \;=\; \sqrt{200{-}20e{+}e^2}\,,$$

$$L \;=\; e + \sqrt{200{-}20e{+}e^2}\,, \qquad T \;=\; e + 2\sqrt{200{-}20e{+}e^2}\,.$$

   The relevant domain is $e \in [0, 10]$.

4. For question (a), the critical points are given by:

$$\frac{dL}{de} \;=\; 1+\tfrac{1}{2}(200-20e+e^2)^{-1/2}(200-20e+e^2)' \;=\; 1-\frac{10-e}{\sqrt{200-20e+e^2}} \;=\; 0\,,^*$$

which reduces to $\sqrt{200-20e+e^2} = 10-e$, then to $200-20e+e^2 = (10-e)^2$, which cancels to the impossible equation $200 = 100$. Thus, there are *no* critical points, and the absolute minimum must be one of the endpoints. Since $L(0) = 10\sqrt{2} \approx 14.1 < L(10) = 20$, the minimum is at $e = 0$.

For question (b), the critical points are given by:

$$\frac{dT}{de} \;=\; 1-\frac{2(10-e)}{\sqrt{200-20e+e^2}} \;=\; 0 \quad\Longrightarrow\quad \sqrt{200-20e+e^2} \;=\; 20-2e$$

$$\Longrightarrow\quad 200-20e+e^2 \;=\; (20-2e)^2 \quad\Longrightarrow\quad 3e^2-60e+200 \;=\; 0\,.$$

The Quadratic Formula then gives:

$$e \;=\; \tfrac{60\pm\sqrt{60^2-4(3)(200)}}{2(3)} \;=\; 10 \pm \tfrac{10}{3}\sqrt{3} \;\approx\; 4.2,\ 15.8\,.$$

The second solution is outside the domain $e \in [0,10]$, so the only relevant critical point is $e = 10-\frac{10}{3}\sqrt{3} \approx 4.2$, with value $T(e) = 10+10\sqrt{3} \approx 27.3\,$. Comparing to endpoints $T(0) = 20\sqrt{2} \approx 28.3$ and $T(10) = 30$, we find the absolute minimum at $e = 10-\frac{10}{3}\sqrt{3} \approx 4.2$ with $T(e) = 10+10\sqrt{3} \approx 27.3$.

5. For question (a), the minimum distance at $e = 0$ is obvious in retrospect: the straight diagonal is the shortest path between opposite corners.

   For question (b), the minimum time is about $T(4.2) \approx 27.3$ sec: that is, at a speed between 0.5 and 1 cm/sec, the ants can cross the 10 cm $\times$ 10 cm square in about 27 sec, which is reasonable. This is a slight saving over the straight diagonal path, which takes about 28 sec. (This assumes they move at carpet speed along the right edge of the square; if they moved at floor speed, they would do much better to go around the carpet, at 20 sec.)

A line of ants will usually find the minimum distance path over a landscape by gradually tightening their curves; what do you think they would do in this case?

Maximizing profit. The Acme Company produces widgets for \$10 each and sells them for s dollars each. The number of widgets sold is modeled by the market demand function $m(s) = 100-s$: for example, if they charge \$25, customers will buy $m(25) = 75$ widgets, but price \$100 is too high for the market: $m(100) = 0$.

PROBLEM: What selling price s will maximize total profits?

The independent variable is $s \in [10,100]$. Profit per widget is $s-10$. Total profit is $P(s) = m(s)\,(s-10) = (100-s)(s-10) = -1000+110s-s^2$. The critical point $P'(s) = 110-2s = 0$ is $s = 55$, which is clearly the maximum point, since $P(55) = 2025$ but the endpoints produce $P(10) = P(100) = 0$. Thus the most profitable selling price is $s = \$55$.

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* Note $\frac{dL}{de}$ is defined over the whole domain $e \in [0,10]$, since $200-20e+e^2 = 10^2+(10-e)^2 > 0$.