Vertical asymptotes. We say a curve has a line as an asymptote if, as the curve runs outward to infinity, it gets closer and closer to the line. “Closer and closer” reminds us of limits, and indeed we have seen that \( x = a \) is a vertical asymptote of \( y = f(x) \) whenever one of the following holds:

\[
\lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^+} f(x) = \infty \quad \lim_{x \to a^-} f(x) = -\infty \quad \lim_{x \to a^+} f(x) = -\infty.
\]

As we saw in §1.5, \( \infty \) has no meaning by itself; rather, the whole equation means that, as \( x \) gets closer to (but unequal to) \( a \), the output \( f(x) \) eventually becomes higher than any given bound \( B \), such as \( B = 100 \) or \( 1000 \) or \( 1 \) billion. Similarly, a limit equals \( -\infty \) when \( f(x) \) becomes lower than \( -B \) for any large \( B \).

At the end of §3.3, we saw how a sign chart for \( f'(x) \) can classify vertical asymptotes. We could do this with a sign chart for \( f(x) \) itself, with no derivatives.

**Example:** Let:

\[
f(x) = \frac{x^2 - 6x + 9}{x^3 - 6x^2 + 11x - 6} = \frac{(x-3)^2}{(x-1)(x-2)(x-3)} = \frac{x-3}{(x-1)(x-2)}.
\]

(To determine vertical asymptotes and intercepts, we always want \( f(x) \) in factored* form.) In the original form, the denominator vanishes at \( x = 3 \), but we work with the cancelled form at right.

The function can only change its sign at points where \( f(x) = 0 \) (numerator = 0) or \( f(x) \) is not defined (denominator = 0), that is, \( x = 1, 2, 3 \). In the interval \( x \in (-\infty, 1) \), the sign is given by a sample point like \( f(0) = \frac{-2}{(-1)(-3)} = -\frac{2}{3} < 0 \), so \( f(x) \) is negative; and similarly for the other intervals.

\[
\begin{array}{c|c|c|c|c}
 x & 1 & 2 & 3 \\
\hline
 f(x) & -\infty & + & - & 0 + \\
\end{array}
\]

Each time \( x \) passes one of the sign-change candidates \( x = a \), a factor \((x-a)\) changes from negative to positive, and \( f(x) \) does indeed change sign.

---

* Notes by Peter Magyar magyar@math.msu.edu

* To factor the bottom, we try linear factors \( x - \frac{m}{n} \), where \( m \) is an integer factor of the constant coefficient \( 6 \), and \( n \) is an integer factor of the highest coefficient 1, so \( n = \pm 1, \pm 2, \pm 3, \pm 6 \) and \( m = \pm 1 \). Trying \( \frac{m}{n} = 1 \), we find \( x - 1 \) is a factor, since polynomial long division gives \( x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) \), and the quadratic is easy to factor. For a review of polynomial long division, see Khan Academy: www.khanacademy.org/math/algebra2/polynomial_and_rational/dividing_polynomials/v/polynomial-division.
Here \( f(x) = \pm \infty \) just means the denominator vanishes and there is a vertical asymptote. The signs on each side of the asymptote show whether the graph shoots upward or downward: we have \( \lim_{x \to 1^-} f(x) = -\infty \), \( \lim_{x \to 1^+} f(x) = \infty \), \( \lim_{x \to 2^-} f(x) = \infty \), \( \lim_{x \to 2^+} f(x) = -\infty \).

**Horizontal asymptotes.** To understand the behavior of the graph over the left and right ends of the \( x \)-axis, we will need a new kind of limit in which \( x \) becomes larger and larger.

**Definition:**

- \( \lim_{x \to \infty} f(x) = L \) means that \( f(x) \) can be forced arbitrarily close to \( L \), closer than any given \( \varepsilon > 0 \), by making \( x > B \) for some \( B \).
- \( \lim_{x \to -\infty} f(x) = L \) means that \( f(x) \) can be forced arbitrarily close to \( L \), closer than any given \( \varepsilon > 0 \), by making \( x < -B \) for some \( B \).

Graphically, \( \lim_{x \to \infty} f(x) = L \) means that toward the right of the \( x \)-axis, the graph \( y = f(x) \) approaches the horizontal asymptote \( y = L \); and similarly for \( \lim_{x \to -\infty} f(x) = L \) toward the left. We can even have \( \lim_{x \to \infty} f(x) = \infty \), which means that the graph goes off toward the upper right of the \( xy \)-plane in an unspecified way.

The most basic \( x \to \infty \) limits are the power functions: for a positive real number power \( p > 0 \), we have:

\[
\lim_{x \to \infty} x^p = \infty, \quad \lim_{x \to \infty} \frac{1}{x^p} = 0.
\]

For \( x \to -\infty \), consider the rational power \( p = \frac{m}{n} \) where \( m, n \) are positive integers with \( n \) odd (perhaps \( n = 1 \)); then:

\[
\lim_{x \to -\infty} x^{m/n} = \begin{cases} \infty & \text{for } m \text{ even} \\ -\infty & \text{for } m \text{ odd} \end{cases}, \quad \lim_{x \to -\infty} \frac{1}{x^{m/n}} = 0.
\]

\( \dagger \) Proof: For any large bound \( C \), we can force \( x^p > C \) if we take \( x \) so large that \( x > C^{1/p} \). For any small error tolerance \( \varepsilon > 0 \), we can force \( |\frac{1}{x^p} - 0| < \varepsilon \) if we take \( x \) so large that \( x > (\frac{1}{\varepsilon})^{1/p} \).
For example:

Based on these, we can deduce the horizontal asymptotes for any rational function (quotient of polynomials).

**EXAMPLE:** Continuing $f(x) = \frac{x^2 - 6x + 9}{x^3 - 6x^2 + 11x - 6}$, does $y = f(x)$ have a horizontal asymptote? Informally, we can reason as follows. For large $x$ (positive or negative), the value of $x^2 - 6x + 9$ is relatively close to $x^2$: say for $x = 1000$, compare $x^2 - 6x + 9 = 9,994,009$ and $x^2 = 1,000,000$. Thus we can approximate $x^2 - 6x + 9 \approx x^2$, which we call the highest term of the polynomial. Also doing this for the denominator:

$$f(x) = \frac{x^2 - 6x + 9}{x^3 - 6x^2 + 11x - 6} \approx \frac{x^2}{x^3} \text{ for large } x.$$

Thus, $\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{x^2}{x^3} = \lim_{x \to \pm \infty} \frac{1}{x} = 0$, and $y = f(x)$ has the horizontal asymptote $y = 0$ for $x \to \infty$ and $x \to -\infty$. In the graph we drew previously, the left and right ends do indeed approach the $x$-axis.

Formally, we can show this from the Limit Laws by dividing numerator and denominator by the highest term in the denominator:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 - 6x + 9}{x^3 - 6x^2 + 11x - 6} = \lim_{x \to \infty} \frac{x^2 - 6x + 9}{x^3 - 6x^2 + 11x - 6} \cdot \frac{1}{x^3}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{6}{x^2} + \frac{9}{x^3}}{1 - \frac{6}{x} + \frac{11}{x^2} - \frac{6}{x^3}} = \frac{0 - 6(0) + 9(0)}{1 - 6(0) + 11(0) - 6(0)} = 0.$$

**Warning:** The informal argument is the easiest way to understand these limits, but the formal argument (dividing by the highest term) might be required for full credit on a quiz or test.

**EXAMPLE:** For $f(x) = \frac{3x^2 - x + 9}{5x^2 + 2x - 6}$, we take highest terms to get:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2 - x + 9}{5x^2 + 2x - 6} = \lim_{x \to \infty} \frac{3x^2}{5x^2} = \frac{3}{5}.$$

Thus, $y = f(x)$ has horizontal asymptote $y = \frac{3}{5}$ toward the right. We similarly deduce $\lim_{x \to -\infty} f(x) = \frac{3}{5}$, which means the same horizontal asymptote toward the left.
EXAMPLE: For

\[ f(x) = \frac{x^2 + 3x^{7/2} - x^{-5}}{9x\sqrt{x} + 4x^2\sqrt{x}}, \]

the terms in the denominator are \(9x^{1/2} = 9x^{3/2}\) and \(4x^2x^{1/2} = 4x^{5/2}\), so the second is the highest term. Thus:

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 + 3x^{7/2} - x^{-5}}{9x\sqrt{x} + 4x^2\sqrt{x}} = \lim_{x \to \infty} \frac{3x^{7/2}}{4x^{5/2}}
\]

\[= \lim_{x \to \infty} \frac{3}{4}x^{7/2-5/2} = \lim_{x \to \infty} \frac{3}{4}x = \infty, \]

which means \(y = f(x)\) has no horizontal asymptote. However, the approximation \(f(x) \approx \frac{3}{4}x\) implies that the right end of the graph looks like a line with slope \(\frac{3}{4}\). (See slant asymptotes in §3.5.) This function is not defined for \(x < 0\), so there is no left end.