Increasing and decreasing functions. We will see how to determine the important features of a graph $y = f(x)$ from the derivatives $f'(x)$ and $f''(x)$, summarizing our Method on the last page. First, we consider where the graph is rising $\nearrow$ and falling $\searrow$. Formally:

> *Definition:* A function $f(x)$ is *increasing* on the interval $[a, b]$ whenever $f(x_1) < f(x_2)$ for every pair of inputs $x_1 < x_2$ in $[a, b]$; and $f(x)$ is *decreasing* on $[a, b]$ whenever $f(x_1) > f(x_2)$ for every $x_1 < x_2$.

We can determine this with derivatives: the graph rises where its slope is positive.

> *Increasing/Decreasing Theorem:* Let $f(x)$ be continuous on $[a, b]$.
> - If $f'(x) > 0$ for all $x \in (a, b)$,* then $f(x)$ is increasing on $[a, b]$.
> - If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on $[a, b]$.

Proof. Assume $f'(x) \geq 0$ for all $x \in (a, b)$, and consider any $x_1 < x_2$ in $[a, b]$. Applying the Mean Value Theorem to the interval $[x_1, x_2]$, we have:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;=\; f'(c) \;>\; 0,$$

since all derivatives $f'(x)$ are positive. Multiplying by $x_2 - x_1 > 0$, we get:

$$f(x_2) - f(x_1) \;>\; (x_2 - x_1)0 \;=\; 0, \quad \text{so } f(x_2) > f(x_1).$$

That is, for all $x_1 < x_2$ in the interval $[a, b]$, we have $f(x_1) < f(x_2)$, and $f(x)$ is increasing. The second statement of the Theorem is proved similarly. Q.E.D.

EXAMPLE: For $f(x) = x^5 - 15x^3$, let us determine the rough shape of the graph by examining the derivative:

$$f'(x) \;=\; 5x^4 - 45x^2 \;=\; 5x^2(x^2 - 9) \;=\; 5x^2(x-3)(x+3).$$

Since $f'(x)$ is defined everywhere, the critical points (or critical numbers) are the solutions of $f'(x) = 0$, namely $x = -3, 0, 3$.

x		-3		0		3	
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$
$f(x)$	$\nearrow$	162	$\searrow$	0	$\searrow$	-162	$\nearrow$

Since $f'(x)$ is zero only at the critical points, it is all positive or all negative in each interval between. For example, in the leftmost interval $(-\infty, 3)$, a sample value is $f'(-4) = 560 > 0$, so $f'(x)$ is positive in the whole interval, and we put $+$ in the first column next to $f'(x)$. The rest of the $f'(x)$ row is similar.

What does this mean for the graph $y = f(x)$? From §3.1, we know the critical points are candidates for local max/mins: hill tops or valley bottoms. Which is which? To the left of $x = -3$, we have $f'(x) > 0$ so $f(x)$ is increasing $\nearrow$; to the

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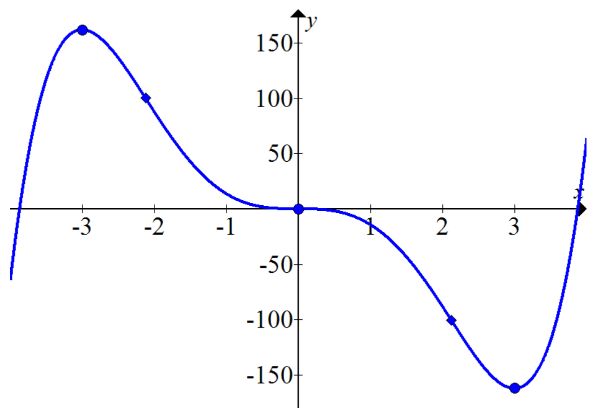
*Recall that $x \in (a, b)$ denotes the open interval, meaning $a < x < b$.

right, we have $f'(x) < 0$ so $f(x)$ is decreasing $\searrow$. Evidently, $x = 3$ is a local max, and the point $(-3, 162)$ is a hill top of the graph. Similarly, $(3, -162)$ is a valley.

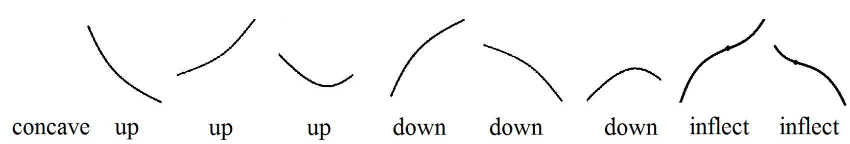
On the other hand, to the left *and* right of $x = 0$, we have $f'(x) < 0$, so $f(x)$ is decreasing on both sides: this means $x = 0$ is a stationary point where the graph levels out before continuing to descend. In fact, $f(x)$ is decreasing for $x \in [-3, 3]$, even though we only have $f'(x) \geq 0$, *not* $f'(x) > 0$ on this interval. We get a good picture of the graph:[†]



The reasoning in our example holds for any function:

> *First Derivative Test:* Let $f(x)$ be a function differentiable in a small interval around $x = c$, with $f'(c) = 0$.
>
> - If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $x = c$ is a local maximum of $f(x)$.
> - If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $x = c$ is a local minimum of $f(x)$.
> - If $f'(x)$ has the same sign on both sides of $x = c$, then $x = c$ is a stationary point of $f(x)$, not an extremal point.

Concavity. A more subtle feature of a graph is where it curves upward or downward. We say a graph is *concave up* near a point if it is part of a smiling curve $\smile$; and *concave down* if it is part of a frowning curve $\frown$. An *inflection point* is a special point where the graph wiggles, changing its concavity: a transition point between smiling and frowning $\sim$.[‡] Some examples:



In terms of the slope, concave up means that as x increases, the slope becomes less negative or more positive. For concave down, the slope becomes less positive or more negative.

[†]Also note that $f(x) = x^5 - 15x^3$ has only odd powers of x, so $f(-x) = -f(x)$. This means the graph has a 180° rotation symmetry, like a propeller. Such an $f(x)$ is called an *odd function.*

[‡]More generally, $y = f(x)$ is concave up if it lies below or on the secant line segment between any two points $(a, f(a))$ and $(b, f(b))$. For example, the absolute value graph $y = |x|$ has a "pointy smile" $\vee$ which lies below every secant line crossing the y-axis, and which contains every secant line on one side of the y-axis, so it is concave up.

Definition: Suppose the derivative $f'(x)$ is defined for x near c./

- $f(x)$ is *concave up* at $x = c$ if $f'(x)$ is *increasing* near $x = c$.
- $f(x)$ is *concave down* at $x = c$ if $f'(x)$ is *decreasing* near $x = c$.
- $f(x)$ has an *inflection point* at $x = c$ if $f'(x)$ has a local max or local min at $x = c$.

We can test for concavity using the second derivative $f''(x)$:

Concavity Theorem: Let $f(x)$ be a function.

- If $f''(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is concave up over (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is concave down over (a, b).
- If $f''(c) = 0$ and $f''(x)$ changes its sign at $x = c$, then $f(x)$ has an inflection point at $x = c$.

Proof. Applying the Increasing/Decreasing Theorem to the function $g(x) = f'(x)$, we get: if $g'(x) > 0$, then $g(x)$ is increasing. But $g'(x) = (f'(x))' = f''(x)$, so this means: if $f''(x) > 0$, then $f'(x)$ is increasing, and $f(x)$ is concave up. The proof of the second part is similar. The third part comes from applying the First Derivative Test to $g(x)$. Q.E.D.

EXAMPLE: Continuing the above $f(x) = x^5 - 15x^3$, $f'(x) = 5x^4 - 45x^2$, we have:

$$f''(x) \;=\; 20x^3 - 90x \;=\; 10x(2x^2 - 9).$$

The candidate inflection points are where $f''(x) = 0$, i.e $x = 0$ and $x = \pm\frac{3}{2}\sqrt{2} \approx \pm 2.12$, and the sign chart confirms the transitions in concavity at these points.

x		$-\frac{3}{2}\sqrt{2}$		0		$\frac{3}{2}\sqrt{2}$	
$f''(x)$	$-\,-$	0	$+\,+$	0	$-\,-$	0	$+\,+$
$f(x)$	$\frown$	$\frac{567}{8}\sqrt{2}$	$\smile$	0	$\frown$	$-\frac{567}{8}\sqrt{2}$	$\smile$

(I wrote double $-\,-$ and $+\,+$ just to make frowny and smiley faces: this is a good way to remember which is which.) This agrees with the features of our graph above, and it allows us to precisely determine the inflection points marked by small diamonds in the picture: $(-\frac{3}{2}\sqrt{2}, \frac{567}{8}\sqrt{2})$, $(\frac{3}{2}\sqrt{2}, -\frac{567}{8}\sqrt{2})$; and also $(0, 0)$, which is both a stationary critical point and an inflection point.

Critical Points and Concavity. There is one more use we can make of the second derivative. At a local max $x = c$, the slope changes from positive to negative, so the graph is concave down and $f''(c) < 0$; while at a local min it is concave up and $f''(x) > 0$. Thus, we can distinguish extremal points just from the sign of $f''(c)$.

Second Derivative Test: Let $f(x)$ be a function with $f''(x)$ continuous near $x = c$. Suppose $f'(c) = 0$.

- If $f''(c) < 0$, then $x = c$ is local maximum of $f(x)$.
- If $f''(c) > 0$, then $x = c$ is local minimum of $f(x)$.
- If $f''(x) = 0$, then this test fails, and $x = c$ might be a local max, a local min, or a stationary point.

Indeed, in our example, we have $f''(-3) = -270 < 0$ at the local max; $f''(0) = 0$ at the stationary point; and $f''(3) = 270 > 0$ at the local min.

Example. We will graph $f(x) = \dfrac{x^{2/3}}{(x-1)^2}$, going through the Method steps on the last page.

1. Using the Quotient and Chain Rules, and much simplification, we get:

$$f'(x) \;=\; \frac{\frac{2}{3}x^{-1/3}(1-x)^2 - x^{2/3}2(x-1)^1(x-1)'}{(1-x)^4} \;=\; -\frac{\frac{2}{3}(2x+1)}{(x-1)^3\,x^{1/3}}\;.$$

$$f''(x) \;=\; \frac{\frac{2}{3}(2)(1-x)^3x^{1/3} - \frac{2}{3}(2x+1)\left(3(x-1)^2x^{1/3} + (x-1)^3\frac{1}{3}x^{-2/3}\right)}{(1-x)^6\,x^{2/3}} \;=\; -\frac{\frac{2}{9}(14x^2+14x-1)}{(x-1)^4\,x^{4/3}}\;.$$

2. The two types of critical points are solutions of:

   - $f'(x) = 0$, when the numerator is zero: $\frac{2}{3}(2x+1) = 0$, i.e. $x = -\frac{1}{2}$
   - $f'(x) =$ undefined, when the denominator is zero, i.e. $x = 1$ and $x = 0$.

3. The sign chart looks like:

x		$-\frac{1}{2}$		0		1	
$f'(x)$	$+$	0	$-$	∞	$+$	∞	$-$
$f(x)$	$\nearrow$	$\frac{2}{9}\sqrt[3]{2}$	$\searrow$	0	$\nearrow$	∞	$\searrow$

   - $(-\frac{1}{2}, \frac{2}{9}\sqrt[3]{2}) \approx (-0.5, 0.28)$ is a local maximum (hill top dot in the pictue below). We could also see this by taking $f''(-\frac{1}{2}) = -\frac{32}{81}\sqrt[3]{2} < 0$.
   - $(0, 0)$ is a local minimum, but instead of a flat valley bottom it is a sharp ravine (a cusp): instead of a horizontal tangent, the slope becomes infinte and the tangent line is vertical.
   - $x = 1$ is a vertical asymptote (dashed line in picture): since the denominator of $f(x)$ is zero, $f(1)$ is undefined and the function blows up to $\pm\infty$. Specifically, $f(x)$ is increasing to the left of $x = 1$, so the graph shoots up to $\lim_{x\to 1^-} f(x) = \infty$; and $f(x)$ is decreasing to the right of $x = 1$, so the graph shoots down from $\lim_{x\to 1^+} f(x) = \infty$.

4. The inflection points are solutions of $f''(x) = 0$, when the numerator is zero:

$$14x^2 + 14x - 1 = 0 \quad\Longleftrightarrow\quad x \;=\; \tfrac{-7\pm 3\sqrt{7}}{14} \;\approx\; -1.07,\ 0.07$$

   These are the small diamond points in the picture.

   Here the solutions of $f''(x) =$ undefined are just the vertical asymptote $x = 1$ of $f(x)$, and also the vertical asymptote $x = 0$ of $f'(x)$. Becasue $f'(a)$ is not defined, these are not considered inflection points, though the concavity does change.

5. The x and y-intercepts are both at $(0, 0)$.

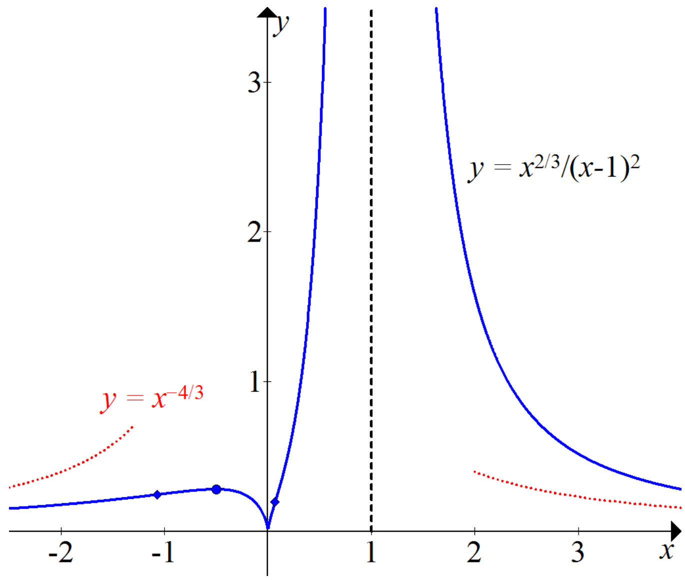
6. When x is a very large positive or negative number, x is almost the same as $x-1$ (compare 1000 and 999). We can approximate $f(x)$ by replacing $x-1$ with x:

$$f(x) \;=\; \frac{x^{2/3}}{(x-1)^2} \;\approx\; \frac{x^{2/3}}{x^2} \;=\; x^{-4/3} \;=\; \frac{1}{x\sqrt[3]{x}} \quad \text{for large } |x|.$$

   This simplified function is easy to graph (dotted curve in the picture), and the true graph $y = f(x)$ approaches this curve like an asymptote at the left and right ends of the x-axis.

7. This function does not have any symmetry.

8. Finally, the graph is:



Method for Graphing

1. Determine the derivatives $f'(x)$ and $f''(x)$.
2. Solve $f'(x) = 0$ and $f'(x) = \text{undef}$ to find the critical points.
3. Sign table: $f'(x) > 0$ means $f(x)$ is $\nearrow$; $f'(x) < 0$ means $f(x)$ is $\searrow$. Classify critical points as: local max, local min, stationary, or vertical asymptote.
4. Solve $f''(x) = 0$ or undef for inflection pts. (Sign table usually not needed.)
5. Find the x-intercepts by solving $f(x) = 0$, and the y-intercept $(0, f(0))$.
6. Find the behavior as $x \to \pm\infty$ by taking the highest-order terms in $f(x)$
7. Check for symmetry: $180°$ rotation symmetry if $f(-x) = -f(x)$; or side-to-side reflection symmetry if $f(-x) = f(x)$.
8. Draw all the above features on the graph.

We will discuss Step 6 in §3.4. A very detailed Method chart is at the end of §3.5.