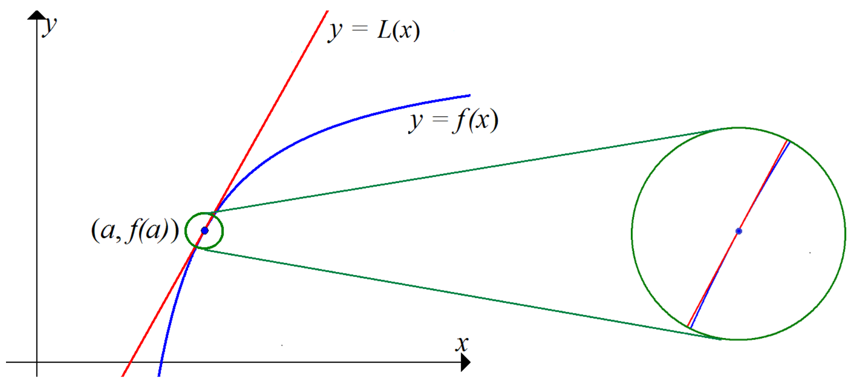
Tangent linear function. The geometric meaning of the derivative $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The tangent line is itself the graph of a linear function $y = L(x)$, where:

$$L(x) \;=\; f(a) + f'(a)(x{-}a).$$

This is correct because the line $y = f(a) + f'(a)(x{-}a)$ has slope $m = f'(a)$, and $L(a) = f(a) + f'(a)(a{-}a) = f(a)$, so the line passes through the point $(a, L(a)) = (a, f(a))$.

The value $f'(a)$ is not just the slope of the tangent line: it is also the slope of the graph itself, because as we zoom in toward $(a, f(a))$, the graph and the tangent line become indistinguishable*:



This suggests a further numerical meaning of the derivative: any function $f(x)$ is very close to being a linear function near a differentiable point $x = a$, so that $L(x)$ is a good approximation for $f(x)$ when x is close to a:

$$f(x) \;\approx\; L(x) \;=\; f(a) + f'(a)(x{-}a) \quad \text{for } x \approx a.$$

Much later in §11.10 of Calculus II, we will study Taylor series, which give much better, higher-order approximations to $f(x)$.

EXAMPLE: Find a quick approximation for $\sqrt{1.1}$ without a calculator. Clearly, this is close to $\sqrt{1} = 1$, but we want more accuracy. Take $f(x) = \sqrt{x}$, so $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(1) = \frac{1}{2}$. For x near $a = 1$, we have the linear function:

$$L(x) \;=\; f(1) + f'(1)(x{-}1) \;=\; 1 + \tfrac{1}{2}(x{-}1),$$

and the linear approximation:

$$\sqrt{1.1} \;=\; f(1.1) \;\cong\; L(1.1) = 1 + \tfrac{1}{2}(0.1) = 1.05.$$

A calculator gives: $\sqrt{1.1} \approx 1.049$, so our answer is correct to 2 decimal places with very little work. Furthermore, we get approximations for all other square roots near 1 for free, for example $\sqrt{0.96} \cong 1 + \frac{1}{2}(0.96{-}1) = 1{-}0.02 = 0.98$.

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*By contrast, if we zoom in toward a non-differentiable point, such as $(0, 0)$ for the graph $y = |x|$, the graph does *not* look more and more linear, but rather keeps its angular appearance.

EXAMPLE: Approximate $\sin(42°)$ without a scientific calculator. This is clearly close to $\sin(45°) = \frac{\sqrt{2}}{2} \approx 0.71$, so let us take $a = 45°$. Now, to use calculus with trig functions, we must always convert to radians: $a = 45(\frac{2\pi}{360}) = \frac{\pi}{4}$ rad. Thus $f'(a) = \sin'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and we have the linear function:

$$L(x) \;=\; \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}(x - \tfrac{\pi}{4}).$$

The linear approximation is:

$$\sin(42°) \;=\; \sin\!\big(42(\tfrac{2\pi}{360})\big) \;\approx\; L(x) \;=\; \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\big(42(\tfrac{2\pi}{360}) - \tfrac{\pi}{4}\big) \;=\; \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\big(\tfrac{\pi}{60}\big) \;\approx\; 0.67.$$

A scientific calculator gives $\sin(42°) \approx 0.669$, so again the linear approximation is accurate to two decimal places.

Input/output sensitivity. We rewrite the linear approximation $f(x) \approx f(a) + f'(a)(x-a)$:

$$\Delta f \;=\; f(x) - f(a) \;\approx\; f'(a)(x-a) \;=\; f'(a)\,\Delta x.$$

This estimates the change of output $f(x)$ away from $f(a)$, in proportion to the change of input x away from a. In Leibnitz notation, with $y = f(x)$, we write:

$$\Delta y \;\approx\; \frac{dy}{dx}\,\Delta x.$$

Here we mean $\frac{dy}{dx} = \frac{dy}{dx}|_{x=a} = f'(a)$. If we think of Δx as an error from an intended input value $x = a$, then $\Delta f \approx f'(a)\,\Delta x$ approximates the error from the intended output $f(a)$.

EXAMPLE: A disk of radius $r = 5$ cm is to be cut from a metal sheet of weight 3 g/cm^2. If the radius is measured to within an error of $\Delta r = \pm 0.2$ cm, what is the approximate range of error in the weight? This is the kind of error-control problem from our limit analyses in Notes §1.7, only now we have the powerful tools of calculus to give a simple answer.

The weight is the density 3 multiplied by the area πr^2, given by the function:

$$W = W(r) = 3\pi r^2 \qquad \text{with} \qquad W(5) = 75\pi \approx 235.6\,,$$

and we aim to find the error ΔW away from this intended value. Since:

$$\tfrac{dW}{dr} \;=\; 3\pi(2r) \;=\; 6\pi r \qquad \text{and} \qquad \tfrac{dW}{dr}|_{r=5} = 30\pi,$$

we have the approximate error:

$$\Delta W \;\approx\; \tfrac{dW}{dr}\Delta r \;=\; 30\pi\,\Delta r.$$

Thus, for $\Delta r = \pm 0.2$, we have $\Delta W \approx 30\pi(0.2) \approx 18.8$. That is:

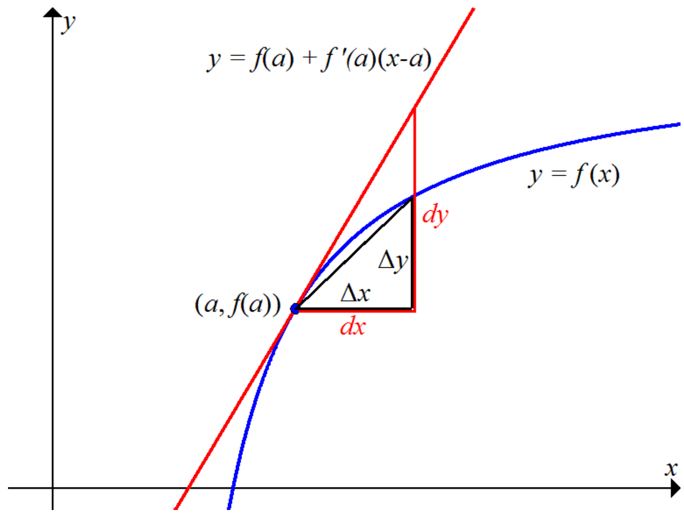
$$r = 5 \pm 0.2 \text{ cm} \qquad \Longrightarrow \qquad W \;\approx\; 235.6 \pm 18.8 \text{ g}\,.$$

The point here is not just the specific error estimate, but the formula which gives, for any small input error Δr, the resulting output error $\Delta W \approx 30\pi\,\Delta r \approx 94\,\Delta r$. The coefficient 30π measures the *sensitivity* of the output W to an error in the input r.

Differential notation. For $y = f(x)$, we rewrite a small Δx as dx, and we define:

$$dy \;=\; \tfrac{dy}{dx}\, dx \qquad \text{and} \qquad df \;=\; f'(x)\, dx.$$

The dependent quantity dy is called a differential: we can think of it as the linear approximation to Δy, as pictured below:



EXAMPLE: We can rewrite the approximation in the previous example as:

$$\Delta W \;\approx\; dW \;=\; \tfrac{dW}{dr}\, dr \;=\; \tfrac{d}{dr}(3\pi r^2)\, dr \;=\; 6\pi r\, dr.$$

Here dr is just another notation for Δr, and the approximation $\Delta W \approx dW = 6\pi r\, dr$ is valid near any particular value of r, such as $r = 5$ in the example.

Linear Approximation Theorem. How close is the approximation $\Delta y \approx dy$, or equivalently $f(x) \approx L(x) = f(a) + f'(a)(x{-}a)$? In fact, the difference between $f(x)$ and $L(x)$ is not only small compared to $\Delta x = x{-}a$, but usually proportional to $(\Delta x)^2 = (x{-}a)^2$, which becomes tiny as $\Delta x \to 0$. (E.g. if $\Delta x = 0.01 = 1\%$, then $(\Delta x)^2 = 0.0001 = 1\%$ of 1%.)

Also, the slower the slope $f'(x)$ changes near $x = a$, the closer $y = f(x)$ is to its tangent line, and this deviation is measured by the rate of change of $f'(x)$, namely the second derivative $f''(x)$. The following theorem gives an upper bound on the error in the linear approximation, $\varepsilon(x) = f(x) - L(x)$.

Theorem: Suppose $f(x)$ is a function such that $|f''(x)| < B$ on the interval $x \in [a{-}\delta, a{+}\delta]$. Then, for all $x \in [a{-}\delta, a{+}\delta]$, we have:

$$f(x) \;=\; f(a) + f'(a)(x{-}a) + \varepsilon(x), \qquad \text{where} \qquad |\varepsilon(x)| < \tfrac{1}{2}B|x{-}a|^2.$$

We give the proof in §3.2 on the Mean Value Theorem.

EXAMPLE: For $f(x) = \sqrt{x}$ near $x = 1$, we have $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(1) = \frac{1}{2}$. Also $f''(x) = -\frac{1}{4}x^{-3/2}$ (a decreasing function), and on the interval $x \in [0.9, 1.1]$, we have:

$$|f''(x)| \le |f''(0.9)| = \tfrac{1}{4}(0.9)^{-3/2} \approx 0.29 < \tfrac{1}{3}.$$

Thus we may take $B = \frac{1}{3}$ and $\frac{1}{2}B = \frac{1}{6}$, so that:

$$\sqrt{x} \;=\; \sqrt{1} + \tfrac{1}{2}(x{-}1) + \varepsilon(x), \qquad \text{where} \qquad |\varepsilon(x)| < \tfrac{1}{6}|x{-}1|^2.$$

For example, the error at $x = 1.1$ is $|\varepsilon(1.1)| < \frac{1}{6}(0.1)^2 < 0.002$, so:

$$\sqrt{1.1} \;=\; 1 + \tfrac{1}{2}(0.1) \pm 0.002 \;=\; 1.05 \pm 0.002\,.$$

Indeed, the calculator value $\sqrt{1.1} \approx 1.049$ lies in the error interval $(0.048, 0.052)$.