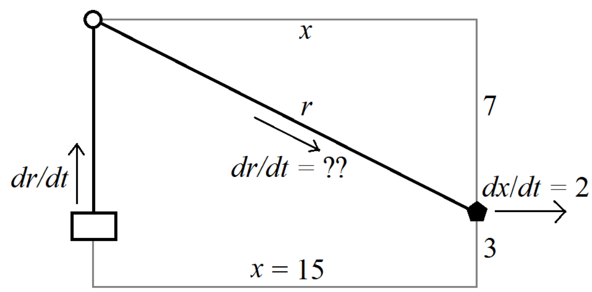
Pulley example. Consider a weight hanging from a rope which stretches up to a pulley 10 ft above the floor, then to your hand, which is 3 ft above the floor and 15 ft horizontally from the pulley. If you walk away from the pulley at 2 ft/sec, how fast will the weight rise?

We want to find an unknown rate of change from a known rate which is related to it geometrically. To start any such problem, we draw a picture and label constant parts with their values: the lengths 3 and 7 below, which will not change as your hand moves horizontally. We label variable parts with letter names: the variable $h = h(t)$ is the horizontal distance from weight to hand, and $r = r(t)$ is the length of rope from pulley to hand, both functions of time t.

The problem specifies the current values of some variables, usually meaning at time $t = 0$: here $h(0) = 15$. Finally, for each variable we draw an arrow marked with its current rate of change: we know $h'(0) = 2$, and $r'(0)$ is the target rate which we aim to compute, since the weight goes upward at the same rate as r increases.



Next, we write equations implied by the geometry of the picture: the Pythagorean Theorem implies $r^2 = h^2 + 7^2$. To determine $r'(0)$, we compute $r(t)$ explicitly, and differentiate:

$$\begin{aligned} r(t) &= \sqrt{h(t)^2+49} \\ r'(t) &= \tfrac{1}{2}(h(t)^2+49)^{-1/2} \cdot \big(h(t)^2+49\big)' \\ &= \tfrac{1}{2}(h(t)^2+49)^{-1/2} \cdot 2h(t)h'(t). \\ &= \frac{h(t)h'(t)}{\sqrt{h(t)^2+49}}\,. \end{aligned}$$

Plugging in the current values at $t = 0$:

$$r'(0) = \frac{h(0)h'(0)}{\sqrt{h(0)^2+49}} = \frac{(15)(2)}{\sqrt{15^2+49}} = \frac{30}{\sqrt{274}} \cong 1.8 \text{ ft/sec}\,.$$

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We could do this a bit more simply by implicitly differentiating both sides of the equation $r^2 = h^2 + 7^2$, then solving for $r'(t)$:

$$\begin{aligned}\big(r(t)^2\big)' &= \big(h(t)^2 + 49\big)' \\ 2r(t)r'(t) &= 2h(t)h'(t) \\ r'(t) &= \frac{h(t)h'(t)}{r(t)}\,.\end{aligned}$$

Now, $r(0) = \sqrt{h(0)^2 + 49} = \sqrt{274}$, so plugging in current values: $r'(0) = \frac{(15)(2)}{\sqrt{274}}$ as before.

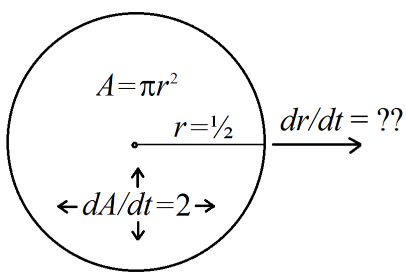
Warning: It is essential to plug in the current values only in the *last step*: if we substituted before differentiating, we would get: $(r(0))' = (\sqrt{h(0)^2 + 49})' = 0$ since the derivative of any constant (even a complicated constant) is *zero*.

Method for related rates problems

1. Draw a picture labeled with:
   - numerical constant values
   - letter variables and their known current values
   - arrows showing known current rates of change (derivatives)
   - an arrow for the unknown rate of change which is desired (the target rate)
2. Write an equation relating the variables according to the geometry of the picture.
3. Assuming each variable is a function of time t, take the derivative $\frac{d}{dt}$ of both sides of the equation, with the Chain Rule producing derivatives of the variables. If necessary, solve the derivative equation for the derivative which is desired.
4. Plug in the current values of the variables and rates to compute the target rate.

Ice block example. We saw a related rates problem in Notes §2.3, last page.

Spill radius example. A stream of water is spreading a circular puddle on the floor. If the puddle is 1 meter across, and the stream increases the area at a rate of 2 sq m/min, then how quickly is the puddle widening?



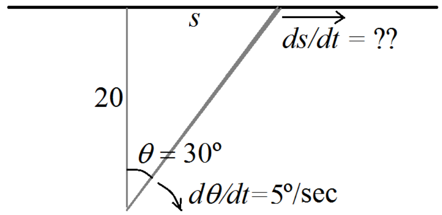
The variable quantities are the radius r and the area A. We know the current value $r(0) = \frac{1}{2}$ and the current rate $A'(0) = \frac{dA}{dt}\big|_{t=0} = 2$. The unknown rate which we must find is $r'(0)$. The area is related to the radius by the equation: $A = \pi r^2$. Differentiating the equation:

$$A'(t) \;=\; \pi\big(r(t)^2\big)' \;=\; 2\pi\, r(t)\, r'(t).$$

Solving for the target rate: $r'(t) = \frac{A'(t)}{2\pi\, r(t)}$, and $r'(0) = \frac{A'(0)}{2\pi\, r(0)} = \frac{2}{2\pi(\frac{1}{2})} = \frac{2}{\pi} \cong 0.64$ m/min.

It is important to check a real-world result for plausibility. The puddle's radius is growing (positive derivative) at a rate of about half a meter per minute, which is reasonable.

Searchlight example: A searchlight is shining along a wall 20 meters away. If the position of the light is 30° away from looking directly at the wall, and the light is turning at 5° per second, then what is the speed of the spotlight image moving along the wall?



The distance from the wall is the constant 20; the variable quantities are θ and s. The angle $\theta(t)$ has current value $\theta(0) = 30°$ and current rate $\theta'(0) = 5°/\mathrm{sec}$, and we seek to compute the unknown rate $s'(0) = \frac{ds}{dt}|_{t=0}$. From the definition of tangent, we have the equation: $\tan(\theta) = \frac{s}{20}$, so we can easily solve for $s = 20\tan(\theta)$. Differentiating (in Leibniz notation this time):

$$\frac{ds}{dt} \;=\; \frac{d}{dt}(20\tan(\theta)) \;=\; 20\sec^2(\theta)\cdot\frac{d\theta}{dt},$$

since $\frac{d}{dx}\tan(x) = \sec^2(x)$ from the table in Notes §2.4. We do not need to solve for $\frac{ds}{dt}$, since we already solved for s before differentiating.

Finally, to plug in the current values of the angles, we must convert them to radians, because the trig differentiation formulas are *only valid for radian measure* (see last page of Notes §2.5). Thus:

$$\theta(0) \;=\; 30° \;=\; 30(\tfrac{2\pi}{360}) \;=\; \tfrac{\pi}{6}\text{ rad},$$

$$\theta'(0) \;=\; \tfrac{d\theta}{dt}|_{t=0} \;=\; 5°/\mathrm{sec} \;=\; 5(\tfrac{2\pi}{360}) \;=\; \tfrac{\pi}{36}\text{ rad/sec},$$

so the current speed is:

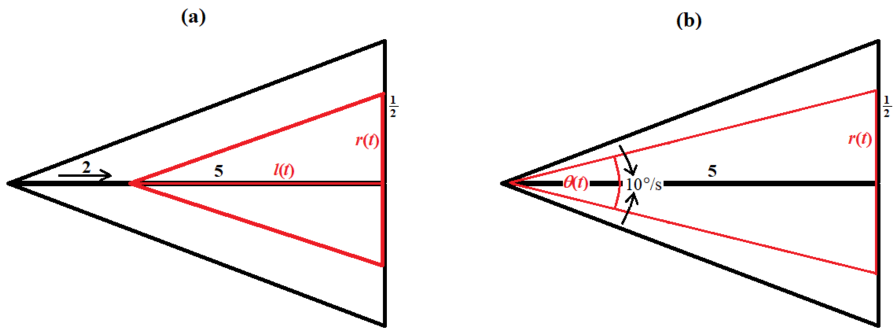
$$s'(0) \;=\; \frac{ds}{dt}\Big|_{t=0} \;=\; 20\sec^2(\tfrac{\pi}{6})\cdot\tfrac{\pi}{36} \;=\; \tfrac{20}{27}\pi \;\cong\; 2.3\text{m/sec}.$$

Note that plugging in $\frac{d\theta}{dt}|_{t=0} = 5$ deg/sec instead of $\frac{\pi}{36} \cong 0.09$ rad/sec would give a wildly incorrect answer for $s'(0)$ in m/sec: the conversion of θ to radians is essential.

One last point: the problem specifies only the speed of θ, not the velocity toward or away from the wall, so we only know $\theta'(0) = \pm\frac{\pi}{36}$, either plus or minus, though in the picture we assumed it was plus. Thus we can only compute $s'(0) = \pm\frac{20}{27}\pi$, but in any case the speed is $|s'(0)| = \frac{20}{27}\pi$.

Flashlight examples. A flashlight shines a cone of light 1 meter across straight toward a wall 5 meters away. What is the rate of change of the lit circular area if: (a) the light moves toward the wall at 2 meters per second; (b) the light stays at 5 meters from the wall, but the focus narrows so that the total angle decreases at 10° per second.

(a) At left is a side view of the light cone shortening:



Mark this with the variables mentioned in the problem (no need to bring in angles): the distance from the wall $\ell(t)$, the area of the circle $A(t)$, and its radius $r(t)$. We are given current values $\ell(0) = 5$, $\ell'(0) = -2$, $r(0) = \frac{1}{2}$, and we must find $A'(0)$.

The picture shows the initial triangle shrinking over time into a similar triangle, so:

$$\frac{r(t)}{\ell(t)} = \frac{1/2}{5} = \frac{1}{10}, \qquad r(t) = \frac{1}{10}\ell(t), \qquad A(t) = \pi r(t)^2 = \frac{\pi}{100}\ell(t)^2.$$

The last equation relates the variable $\ell(t)$ with the given rate to the target variable $A(t)$. Taking derivatives of both sides gives a relation between the given rate and the target rate:

$$A'(t) = \frac{\pi}{50}\ell(t)\,\ell'(t), \qquad A'(0) = \frac{\pi}{50}\ell(0)\,\ell'(0) = \frac{\pi}{50}(5)(2) = \frac{\pi}{5}\ \mathrm{m^2/s}\,.$$

But wait: the problem did not ask for the speed, which is always positive, but for the rate of change, which includes a sign. Thus, we should write the decreasing distance as $\ell'(0) = -2$, so that $A'(0) = -\frac{\pi}{5} \approx -0.63\ \mathrm{m^2/s}$, which is reasonable compared to $A(0) \approx 0.78$.

(b) At right is a side view of the light cone narrowing. Here $\ell(t) = 5$ is constant, so we don't need to mark it as a variable. Rather, we have the angle $\theta(t)$, the radius $r(t)$, and the target variable $A(t)$. We are given current values $\theta'(0) = -10°/\mathrm{sec}$, $r(0) = \frac{1}{2}$.

Relate angle $\theta(t)$ to distance $r(t)$ using the right triangle with angle $\frac{\theta(t)}{2}$, sides $r(t), 5$:

$$\tan(\tfrac{\theta(t)}{2}) = \tfrac{r(t)}{5}, \qquad r(t) = 5\tan(\tfrac{\theta(t)}{2}), \qquad A(t) = \pi r(t)^2 = 25\pi\tan^2(\tfrac{\theta(t)}{2}).$$

To relate the target rate to the given rate, we take derivatives using the Chain Rule twice:

$$A'(t) \;=\; 50\pi\tan(\tfrac{\theta(t)}{2})(\tan(\tfrac{\theta(t)}{2}))' \;=\; 50\pi\tan(\tfrac{\theta(t)}{2})\sec^2(\tfrac{\theta(t)}{2})\,\tfrac{\theta'(t)}{2}.$$

We must change angles into radians: $\frac{\theta'(0)}{2} = -5°/\mathrm{sec} = -\frac{2\pi}{360}(5)\,\mathrm{rad/sec} = -\frac{\pi}{36}$. We are not given $\frac{\theta(0)}{2}$, but we only need its tan and sec values, which we can deduce from the given right triangle having sides $\frac{1}{2}, 5$ and hypotenuse $\sqrt{(\frac{1}{2})^2 + 5^2} = \sqrt{\frac{101}{4}}$:

$$\tan(\tfrac{\theta(0)}{2}) = \frac{\text{opp}}{\text{adj}} = \frac{1/2}{5} = \frac{1}{10}, \qquad \sec^2(\tfrac{\theta(0)}{2}) = \frac{1}{\cos^2(\frac{\theta(0)}{2})} = \frac{\text{hyp}^2}{\text{adj}^2} = \frac{101/4}{25} = \frac{101}{100}.$$

Finally, $A'(0) = 50\pi(\frac{1}{10})(\frac{101}{100})(-\frac{\pi}{36}) = -\frac{101}{720}\pi^2 \approx -1.38\ \mathrm{m^2/s}$, compared to $A(0) \approx 0.78\ \mathrm{m^2}$. This is reasonable, comparing $\theta'(0) = -10°/\mathrm{sec}$ to $\theta(0) = 2\arctan(\frac{1}{10}) \approx 11°$.