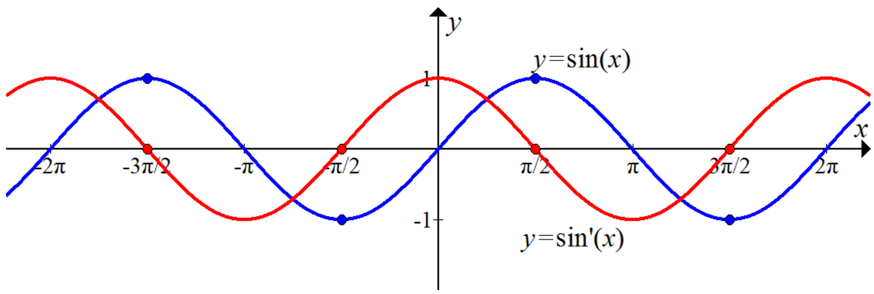
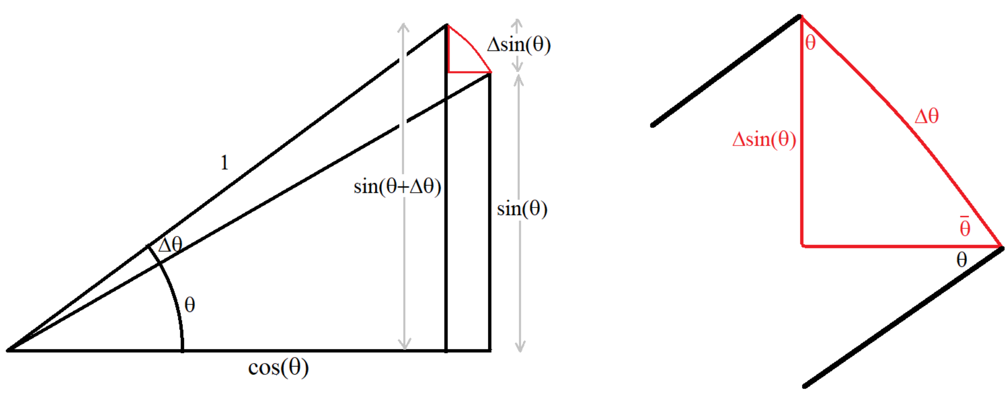
Derivative of sine and cosine. The sine and cosine are important functions describing periodic motion. From the graph $y = \sin(x)$ (in blue), let us examine the slope at each point to sketch the graph of the derivative $y = \sin'(x)$ (in red), as in Notes §2.3:



The graph $y = \sin(x)$ has hills and valleys at $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$, so $\sin'(x) = 0$ at these points. For the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, the slope of $y = \sin(x)$ is positive with a steepest slope of about 1 at $x = 0$, so $y = \sin'(x)$ swells above the x-axis from 0 to 1 to 0, and similarly on the other intervals. The graph we have drawn seems to be roughly the cosine function, so we may guess that $\sin'(x) \overset{??}{=} \cos(x)$. In fact, this is true:

THEOREM: $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

Proof: Here is Newton's original geometric argument. Consider as below a right triangle with hypotenuse 1, angle $x = \theta$, and height $\sin(\theta)$; and another triangle with slightly larger angle $\theta + \Delta\theta$ and slightly larger height $\sin(\theta{+}\Delta\theta)$.



The small red triangle is enlarged at right. It is roughly a right triangle, but its hypotenuse is a slightly curved arc of the unit circle with length $\Delta\theta$, since radian angle measures arclength. Its height is $\Delta\sin(\theta) = \sin(\theta{+}\Delta\theta) - \sin(\theta)$. By equality of alternate interior angles, the angle below the red triangle is θ, and the red triangle has one angle $\bar{\theta} = \frac{\pi}{2} - \theta$ and the other approximately θ. Thus we can approximate the cosine of θ as the red side

Notes by Peter Magyar `magyar@math.msu.edu` adjacent to θ divided by the hypotenuse:

$$\cos(\theta) \;\approx\; \frac{\Delta\sin(\theta)}{\Delta\theta} \;=\; \frac{\sin(\theta{+}\Delta\theta) - \sin(\theta)}{\Delta\theta}.$$

As the angle increment becomes very small, $h = \Delta\theta \to 0$, the circular arc becomes more and more straight, and the approximation becomes an equality in the limit.

$$\cos(\theta) \;=\; \lim_{\Delta\theta\to 0} \frac{\sin(\theta{+}\Delta\theta) - \sin(\theta)}{\Delta\theta} \;\stackrel{\text{def}}{=}\; \sin'(\theta).$$

We can show $\cos'(\theta) = -\sin(\theta)$ by examining the horizontal side of the small triangle, or by using $\cos(\theta) = \sin(\frac{\pi}{2}{-}\theta)$.

To be precise, we give error bounds. The exact angles inside the red sector are $\bar{\theta}$ and $\theta{+}\Delta\theta$. The straight line hypotenuse (secant of the sector) has length $h_1 = 2\sin(\Delta\theta/2) < \Delta\theta$, forming a right triangle with angles $\theta_1 < \theta{+}\Delta\theta$ and $\bar{\theta}_1 < \bar{\theta}$, so that $\theta < \theta_1$. Thus:

$$\cos(\theta) > \cos(\theta_1) = \frac{\Delta\sin(\theta)}{h_1} > \frac{\Delta\sin(\theta)}{\Delta\theta}.$$

On the other hand, if we draw a tangent from the upper vertex to the horizontal red line, we get a right triangle with angle $\theta{+}\Delta\theta$ and hypotenuse $h_2 > \tan(\Delta\theta) > \Delta\theta$. Hence:

$$\frac{\Delta\sin(\theta)}{\Delta\theta} > \frac{\Delta\sin(\theta)}{h_2} = \cos(\theta{+}\Delta\theta).$$

Thus $\frac{\Delta\sin(\theta)}{\Delta\theta}$ is squeezed between $\cos(\theta)$ and $\cos(\theta{+}\Delta\theta)$, and the limit follows.

Here we used $\sin(\Delta\theta) < \Delta\theta < \tan(\Delta\theta)$. To show this geometrically, compare areas of three increasing regions with angle $\Delta\theta$: secant isosceles triangle $\frac{1}{2}\sin(\Delta\theta)$; sector $\frac{1}{2}\Delta\theta$; tangent right triangle $\frac{1}{2}\tan(\Delta\theta)$.

COROLLARY: (a) $\displaystyle\lim_{h\to 0} \frac{\sin(h)}{h} \;=\; 1$ (b) $\displaystyle\lim_{h\to 0} \frac{\cos(h)-1}{h} \;=\; 0.$

Proof: The first limit is just the derivative of sine at zero:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = \lim_{h\to 0} \frac{\sin(0{+}h) - \sin(0)}{h} = \sin'(0) = \cos(0) = 1,$$

and similarly for the second. (Or squeeze using $\sin(h) < h < \sin(h)/\cos(h)$, proved above.)

General trigonometric derivatives. From these basic derivatives, we can compute the derivative of any trig function or combination of trig functions.

EXAMPLE: Compute the derivative of $\tan(x)$. By the Quotient Rule for derivatives (§2.3):

$$\tan'(x) \;=\; \left(\frac{\sin(x)}{\cos(x)}\right)' \;=\; \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos^2(x)}$$

$$=\; \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \;=\; \frac{1}{\cos^2(x)} \;=\; \sec^2(x),$$

since $\cos^2(x) + \sin^2(x) = 1$. In fact, we get the following derivatives:

$f(x)$	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\sec(x)$	$\csc(x)$	$\cot(x)$
$f'(x)$	$\cos(x)$	$-\sin(x)$	$\sec^2(x)$	$\tan(x)\sec(x)$	$-\cot(x)\csc(x)$	$-\csc^2(x)$

Warning: These formulas are for angle x in **radians**, NOT in degrees (see §2.5 end).

Limits of quotients. We can also compute trigonometric limits of the form $\frac{0}{0}$. The trick is to manipulate the numerators and denominators to get factors of the form $\frac{\sin(g(x))}{g(x)}$, where $g(x)$ is any quantity which goes to zero.

EXAMPLE: Compute $\lim_{x\to 0} \frac{\sin(3x)}{x}$. We have:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} \;=\; \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{3x}{x} \;=\; \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \lim_{x\to 0} \frac{3x}{x} \;=\; \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \lim_{x\to 0} 3 \;=\; 1 \cdot 3 \;=\; 3.$$

Here we use $\lim_{x\to 0} \frac{\sin(3x)}{3x} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$, where we substitute* $h = g(x) = 3x$, so that $x \to 0$ forces $h \to 0$.

EXAMPLE: Compute $\lim_{x\to 0} \frac{\tan(x)}{\sin(\sqrt{x})}$. Starting with $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we get:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)}{\sin(\sqrt{x})} \;&=\; \lim_{x\to 0} \frac{1}{\cos(x)} \cdot \sin(x) \cdot \frac{1}{\sin(\sqrt{x})} \\
&=\; \lim_{x\to 0} \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{x} \cdot x \cdot \frac{\sqrt{x}}{\sin(\sqrt{x})} \cdot \frac{1}{\sqrt{x}} \\
&=\; \lim_{x\to 0} \frac{\sqrt{x}}{\cos(x)} \cdot \frac{\sin(x)}{x} \cdot \frac{1}{\frac{\sin(\sqrt{x})}{\sqrt{x}}} \;=\; \frac{\sqrt{0}}{\cos(0)} \cdot 1 \cdot \frac{1}{1} \;=\; 0\,,
\end{aligned}$$

where $\lim_{x\to 0} \frac{\sin(\sqrt{x})}{\sqrt{x}} = 1$ by the substitution $h = g(x) = \sqrt{x}$.

*By the Limit Substitution Theorem at the end of Notes §1.7.