Derivative Function

In Notes §2.1, we defined the derivative of a function $f(x)$ at $x = a$, namely the number $f'(a)$. Since this gives an output $f'(a)$ for any input a, the derivative defines a function.

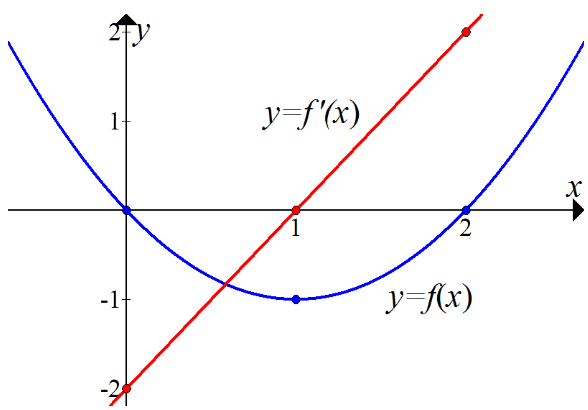
> *Definition:* For a function $f(x)$, we define the *derivative function* $f'(x)$ by:
>
> $$f'(x) \;=\; \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \;=\; \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\,.$$
>
> If the limit $f'(a)$ exists for a given $x = a$, we say $f(x)$ is *differentiable* at a; otherwise $f'(a)$ is undefined, and $f(x)$ is *non-differentiable* or *singular* at a.

This just repeats the definitions in Notes §2.1, except that we think of the derivative as a function of the variable x, rather than as a numerical value at a particular point $x = a$. The choice of letters is meant to suggest different kinds of variables, but they do not have any strict logical meaning: for example, $f(x) = x^2$, $f(a) = a^2$, and $f(t) = t^2$ all define the same function, and $\lim_{x\to a} f(x) = \lim_{t\to a} f(t) = \lim_{z\to a} f(z)$ are all the same limit.

Differentiation. Another name for derivative is *differential.* When we compute $f'(x)$, we *differentiate* $f(x)$. The process of finding derivatives is *differentiation.*

As usual for mathematical objects, we can think of derivatives on four levels of meaning. The physical meaning of $f'(x)$ is the rate of change of $f(x)$ per unit change in x; for example velocity is the derivative of the position function at time t. At the end of Notes §2.1, we also saw how to compute a numerical approximation of a derivative as the diffference quotient for a small value of h (see also §2.9). In this section, we explore the geometric meaning as the slopes of the graph $y = f(x)$, and algebraic methods for computing the limit $f'(x)$.

EXAMPLE: Let $f(x) = x(x-2)$, with graph $y = f(x)$ in blue:



We can sketch the derivative graph $y = f'(x)$ in red, purely from the original graph $y = f(x)$, without any computation. The *slope* of the original graph above a given x-value is the *height* of the derivative graph above that x-value.

At the minimum $x = 1$, the original graph $y = f(x)$ is horizontal and its slope is zero, so $f'(1) = 0$, and we plot the point $(1, 0)$ on the derivative graph $y = f'(x)$. To the right of this point, $y = f(x)$ has positive slope, getting steeper and steeper; so $y = f'(x) > 0$ is above the x-axis, getting higher and higher. Above $x = 2$, the tangent of $y = f(x)$ has slope approximately 2 (considering the relative x and y scales), so we plot $(2, 2)$ on $y = f'(x)$.

Notes by Peter Magyar `magyar@math.msu.edu`

As we move left from $x = 1$, the graph $y = f(x)$ has negative slope, getting steeper and steeper, so $y = f'(x) < 0$ is below the x-axis, getting lower and lower. Above $x = 0$, we estimate $y = f(x)$ to have slope -2, and we plot $(0, -2)$ on $y = f'(x)$. Thus, $y = f'(x)$ looks like the red line in the above picture.

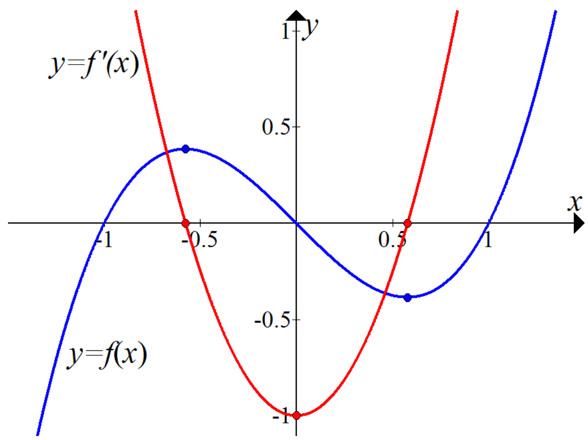
Next we differentiate algebraically. For any value of x:

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x{+}h) - f(x)}{h} \;=\; \lim_{h\to 0} \frac{(x{+}h)(x{+}h{-}2) - x(x{-}2)}{h}$$

$$= \lim_{h\to 0} \frac{x^2{+}2xh{+}h^2{-}2x{-}2h{-}x^2{+}2x}{h} \;=\; \lim_{h\to 0} \frac{2xh{-}2h{+}h^2}{h} \;=\; \lim_{h\to 0} 2x{-}2{+}h \;=\; 2x - 2.$$

That is, $f'(x) = 2x - 2$, which agrees with our sketch of the derivative graph.

EXAMPLE: Let $f(x) = x^3 - x$, with graph in blue:



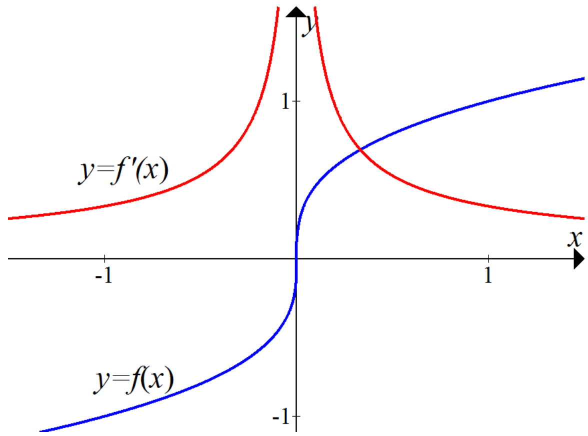
The original graph $y = f(x)$ has a valley with horizontal tangent at $x \cong 0.6$, so the derivative $f'(0.6) \cong 0$, and we plot the approximate point $(0.6, 0)$ on the derivative graph $y = f'(x)$; and similarly the hill on $y = f(x)$ corresponds to the point $(-0.6, 0)$ on $y = f'(x)$. Between these x-values, the slope of $y = f(x)$ is negative, with the slope at $x = 0$ being about -1, so $y = f'(x) < 0$ is below the x-axis, bottoming out at $(0, -1)$.

Algebraically:

$$f'(x) \;=\; \lim_{h\to 0} \frac{((x{+}h)^3{-}(x{+}h)) - (x^3{-}x)}{h} \;=\; \lim_{h\to 0} \frac{(x^3{+}3x^2h{+}3xh^2{+}h^3{-}x{-}h) - x^3{+}x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \;=\; \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 \;=\; 3x^2 - 1.$$

EXAMPLE: Let $f(x) = \sqrt[3]{x}$, the cube root function, with graph in blue:



The slopes of the original graph $y = f(x)$ are all positive, with the same slope above a given x and its reflection $-x$. Thus the derivative graph $y = f'(x) > 0$ lies above the x-axis, and it is symmetric across the y-axis (an even function). The slope of $y = f(x)$ gets smaller for large positive or negative x, and it gets steeper and steeper near the origin, with a vertical tangent at $x = 0$. Thus $y = f'(x)$ approaches the x-axis for large x, and shoots up the y-axis on both sides of $x = 0$, with $f'(0)$ undefined.

Algebraically, we have: $f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$. We must liberate $\sqrt[3]{x+h}$ from under the $\sqrt[3]{}$, so as to be able to cancel $\frac{h}{h}$. In Notes §2.1, we multiplied top and bottom by the conjugate radical, exploiting the identity $(a-b)(a+b) = a^2 - b^2$. Here we have cube roots, so we use the identity: $(a-b)(a^2+ab+b^2) = a^3 - b^3$, taking $a = \sqrt[3]{x+h}$ and $b = \sqrt[3]{x}$:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{x+h}^3 - \sqrt[3]{x}^3}{h(\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2)} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2} = \frac{1}{\sqrt[3]{x+0}^2 + \sqrt[3]{x+0}\,\sqrt[3]{x} + \sqrt[3]{x}^2} = \frac{1}{3\sqrt[3]{x}^2}.$$

In the Notes §2.3, we will develop standard rules for computing derivatives, which let us avoid such complicated limit calculations.

Continuity Theorem. Here is a basic fact relating derivatives and continuity:

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Turing this around, we have the equivalent negative statement (the contrapositive): If $f(x)$ is *not* continuous at $x = a$, then it is *not* differentiable at $x = a$. That is, a discontinuity is also a non-differentiable point (a singularity).

Proof of Theorem: Assume $f(x)$ is differentiable at $x = a$, meaning $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is defined. The Limit Law for Products gives:

$$\lim_{h\to 0}(f(a+h) - f(a)) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot h = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h\to 0} h = f'(a)\cdot 0 = 0.$$

Thus $0 = \lim_{h\to 0}[f(a+h) - f(a)] = [\lim_{h\to 0} f(a+h)] - f(a)$, and $\lim_{h\to 0} f(a+h) = f(a)$, showing that $f(x)$ is continuous at $x = a$.