Derivatives

Definition of derivative. In Notes §1.4, we saw that instantaneous velocity can be obtained as a limit of average velocities over shorter and shorter time increments. Also, the tangent slope of a graph at a given point can be obtained as a limit of secant slopes getting closer and closer to the point. Both these definitions compute a rate of change: velocity is the rate of change of position with respect to time, and slope is the rate of vertical rise with respect to horizontal run.

For any function $f(x)$, we can compute its instantaneous rate of change with respect to x in analogy with the above examples.

Definition. The *derivative* of a function $f(x)$ at $x = a$, denoted $f'(a)$, means:

$$f'(a) \;=\; \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\,.$$

Here $f(a+h) - f(a)$ is the change in $f(x)$ from $x = a$ to $x = a+h$, and h is the increment of x. The average rate of change over the interval* $[a, a+h]$ is the *difference quotient* $\frac{f(a+h)-f(a)}{h}$, and the instantaneous rate of change at $x = a$ is the limit over smaller and smaller increments, $h \to 0$.

Another way to write this is to substitute x for the endpoint of the interval, $a + h = x$, approaching a with increment $h = x - a$:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}\,.$$

In graphical terms, the derivative $f'(a)$ is the slope of the tangent line which touches the graph $y = f(x)$ at the point $(a, f(a))$, and the equation of the tangent line is $y = f(a) + f'(a)(x - a)$.

When the limit $f'(a)$ exists, we say $f(x)$ is *differentiable* at $x = a$. When the limit does not exist, $f'(a)$ is undefined, and we say $f(x)$ is *non-differentiable* or *singular* at $x = a$. In this case, the function $f(x)$ does not have a well-defined rate of change at $x = a$, and the graph $y = f(x)$ does not have a single tangent line at $(a, f(a))$. (See below, *Left and right derivatives.*)

Derivatives of standard functions. A derivative is the limit of a small change in $f(x)$ divided by a small change in x, so it will always be a difficult limit of the form $\frac{0}{0}$, with no defined value if we plug in $h = 0$. To evaluate it, we must find some trick to cancel vanishing factors in the numerator and denominator, as we have seen in Notes §1.4 and §1.6 (*Limits by canceling*).

EXAMPLE: Find $f'(2)$ for $f(x) = \frac{1}{x+1}$. We compute the derivative by combining fractions over a common denominator, then canceling the vanishing factors $\frac{h}{h}$:

$$f'(2) \;=\; \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \;=\; \lim_{h\to 0} \frac{\frac{1}{(2+h)+1} - \frac{1}{2+1}}{h} \;=\; \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{h+3} - \frac{1}{3}\right)$$

Notes by Peter Magyar `magyar@math.msu.edu`

*Note that h can also be a small negative value, in which case this is the rate of change over $[a+h, a]$.

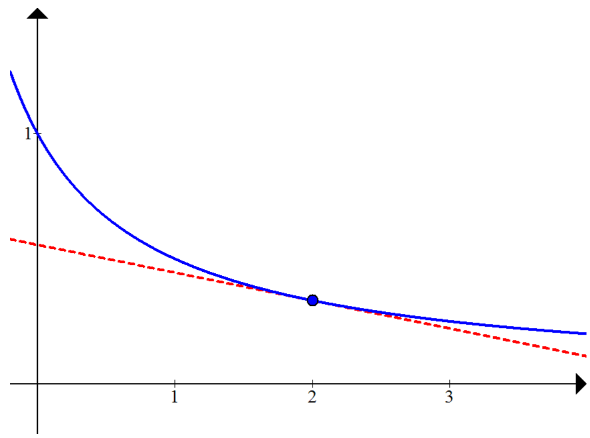
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3-(h+3)}{3(h+3)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{3(h+3)}\right) = \lim_{h\to 0} \frac{-1}{3(h+3)} = \frac{-1}{3(0+3)} = -\frac{1}{9}.$$

Let us compare the same calculation with the alternative variable $x = a + h$:

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x+1}-\frac{1}{2+1}}{x-2} = \lim_{x\to 2} \frac{\frac{3-(x+1)}{3(x+1)}}{x-2}$$

$$= \lim_{x\to 2} \frac{2-x}{3(x+1)(x-2)} = \lim_{x\to 2} \frac{-1}{3(x+1)} = -\frac{1}{3(2+1)} = -\frac{1}{9}\,,$$

where we cancel the vanishing factors $\frac{2-x}{x-2} = \frac{-(x-2)}{x-2} = -1$. Graphically, this looks like:



The curve is $y = f(x) = \frac{1}{x+1}$, and the tangent line at $(a, f(a)) = (2, \frac{1}{3})$ has slope $f'(2) = -\frac{1}{9}$, so its equation is: $y = \frac{1}{3} - \frac{1}{9}(x-2)$.

EXAMPLE: Find $f'(2)$ for $f(x) = \sqrt{x}$. Our trick is to multiply by a conjugate radical to liberate $2+h$ from under the $\sqrt{\ }$, then cancel $\frac{h}{h}$:

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h} = \lim_{h\to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$$

$$= \lim_{h\to 0} \frac{\sqrt{2+h}^{\,2}-\sqrt{2}^{\,2}}{h(\sqrt{2+h}+\sqrt{2})} = \lim_{h\to 0} \frac{h}{h(\sqrt{2+h}+\sqrt{2})} = \frac{1}{\sqrt{2+0}+\sqrt{2}} = \frac{1}{2\sqrt{2}}\,.$$

Here we used the identity $(a-b)(a+b) = a^2 - b^2$ with $a = \sqrt{2+h}$ and $b = \sqrt{2}$.

Left and right derivatives. Let us find $f'(1)$ for the function defined by:

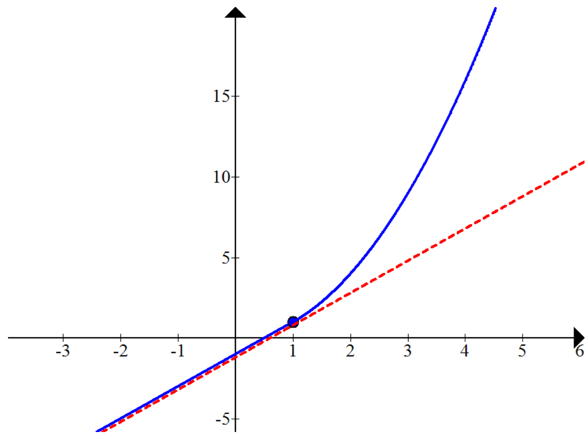
$$f(x) = \begin{cases} 2x-1 & \text{for } x \le 1 \\ x^2 & \text{for } x \ge 1. \end{cases}$$

Since the function is defined differently on the two sides of $x = 1$, we must compute one-sided derivative limits, to see if the two-sided limit exists.

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0^+} \frac{(1+2h+h^2)-1}{h} = \lim_{h\to 0^+} \frac{h(2+h)}{h} = 2\,.$$

$$\lim_{h\to 0^-} \frac{f(1{+}h) - f(1)}{h} \;=\; \lim_{h\to 0^-} \frac{2(1{+}h){-}1 - 1}{h} \;=\; \lim_{h\to 0^-} \frac{2 + 2h - 2}{h} \;=\; 2\,.$$

Since these one-sided limits agree, we have $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = 2$. The graph is:

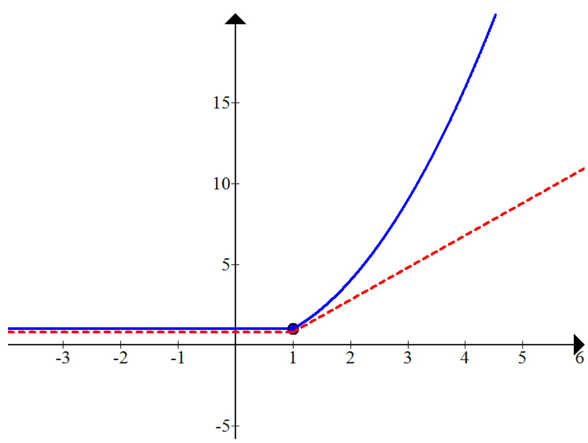


The graph clearly has a transition at $x = 1$, but it still has a well-defined tangent line.

On the other hand, if we take:

$$g(x) \;=\; \begin{cases} 1 & \text{for } x \le 1 \\ x^2 & \text{for } x \ge 1 \end{cases}$$

Then we can compute that $\lim_{h\to 0^+} \frac{g(1+h)-g(1)}{h} = 2$, but $\lim_{h\to 0^-} \frac{g(1+h)-g(1)}{h} = 0$, which means the graph has two different tangent slopes to the left and right of this point, namely it has a corner:



That is, the derivative $g'(1)$ does not exist, $g(x)$ is non-differentiable at $x = 1$, and the graph does not have a well-defined tangent line at the corner. In real-world terms, this function could model the distance fallen by an object held still, then thrown down with speed 2 at time $t = 1$. Before dropping, the speed is 0; immediately after, the speed is 2; and there is no well-defined speed at the moment of throwing. (A more detailed analysis would take into account the gradual acceleration during the throw, which would round off the corner of the graph.)

Numerical derivatives. In Notes §1.4, we computed instantaneous velocity as the derivative of the position function $f(t)$ with respect to time t. For any function which models the dependence between two real-world variables, the derivative gives the rate of change of the dependent variable with respect to the independent variable.

EXAMPLE: A rough model of atmospheric pressure P at height s is given by the function: $P = f(s) = 15c^s$, where P is in pounds per square inch (psi), s is miles above sea level, and the constant $c = 0.81$. How quickly does the pressure drop, with respect to height, at sea-level and at 2 miles up?

At sea level $s = 0$ ft, the pressure is $f(0) = 15$ psi (about half the pressure of a car tire), and the rate of change (psi of pressure change per mile of height upward) is the derivative:

$$f'(0) \;=\; \lim_{h\to 0} \frac{15c^{0+h} - 15c^0}{h} \;=\; \lim_{h\to 0} 15\,\frac{c^h - 1}{h}\,.$$

In this case, we have no algebraic trick to cancel vanishing factors, so we must be content with a numerical approximation of the difference quotient. (Since $P = f(s)$ is only an approximate model anyway, we lose nothing from this further approximation.)

h	0.1	0.01	0.001	0.0001
$15(c^h{-}1)/h$	-2.85	-3.13	-3.16	-3.16

Thus, $f'(0) \approx -3.16$ psi/mi. This is a negative rate of change because a rise in height gives a drop in pressure. For each mile upward, the pressure decreases by approximately 3.16 psi, or more accurately, a small rise like 0.1 mi would decrease pressure by about 0.316 psi.

Now at $s = 2$ mi, pressure is about $f(2) = 9.84$ psi, and we compute the rate of change:

$$f'(2) \;=\; \lim_{h\to 0} \frac{15c^{2+h} - 15c^2}{h} \;=\; \lim_{h\to 0} \frac{15c^2c^h - 15c^2}{h} \;=\; \lim_{h\to 0} 15c^2\!\cdot\!\frac{c^h - 1}{h} \;=\; c^2 \lim_{h\to 0} 15\,\frac{c^h - 1}{h}\,.$$

Now, $c^2 \approx 0.66$, and the second factor is the limit we approximated before. Thus, $f'(2) \approx (0.66)(-3.16) \approx -2.07$ psi/mi. That is, at an altitude of 2 mi, every 0.1 mi rise decreases the pressure by about 0.207 psi.