Math 132 # Continuity **Stewart §1.8**

One of the most basic features of a function is whether it is *continuous*. Roughly, this means that a small change in x always leads to a fairly small change in $f(x)$, without instantaneous jumps. In physical terms, the position of a particle moving in space is continuous, but the position displayed in a video could have a gap, making the position function jump discontinuously. This can be made precise by saying that near $x = a$, the limit of $f(x)$ is $f(a)$:

Definition: A function $f(x)$ is continuous at $x = a$ whenever $\lim_{x\to a} f(x) = f(a)$.

Graphically, a function is continuous whenever the graph $y = f(x)$ proceeds through the point $(a, f(a))$ without jumps or holes.

Types of discontinuity. If $f(x)$ is defined near $x = a$, continuity can fail in several ways:

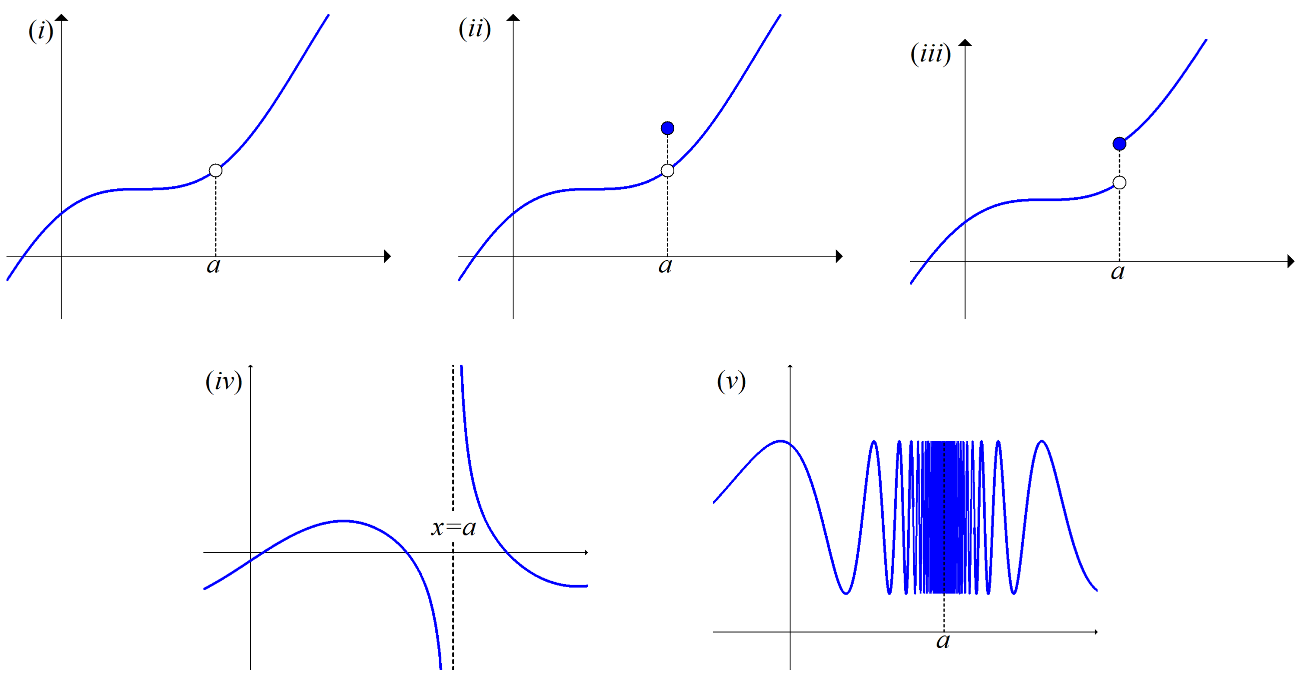
i. Removable discontinuity: $f(a)$ is undefined, but $\lim_{x\to a} f(x)$ exists.

ii. Removable discontinuity: $f(a)$ and $\lim_{x\to a} f(x)$ exist, but are unequal.

iii. Jump discontinuity: the left and right limits are unequal, $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$.

iv. Vertical asymptote: $\lim_{x\to a^+} f(x)$ and/or $\lim_{x\to a^-} f(x)$ are $\pm\infty$.

v. Essential discontinuity: $\lim_{x\to a^+} f(x)$ and/or $\lim_{x\to a^-} f(x)$ do not exist.



We say $f(x)$ is *continuous on an interval* whenever it is continuous at each point of the interval. For the endpoints of a closed interval* $x \in [a, b]$, we cannot take two-sided limits within the interval, so we only require $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

Notes by Peter Magyar `magyar@math.msu.edu`

*The interval $[a, b]$ is the *set* or collection of all numbers x between a and b, including the endpoints. The notation $x \in [a, b]$ means x is an *element* of the set $[a, b]$, meaning it is one of the numbers between a and b, which means $a \leq x \leq b$.

Continuity by cases. For a function defined by cases, whose graph consists of several continuous pieces, the function is continuous provided the pieces join together at the transition points where cases meet.

For example, suppose a weight is winched from the ground at a constant speed for 8 seconds, is dropped, and lands 2 sec later. How fast should the winch haul upward?

The height $s(t)$ feet at t sec is given by:

$$s(t) \;=\; \begin{cases} vt & \text{for } t \in [0,8) \\ 64 - 16(t{-}8)^2 & \text{for } t \in [8,10]. \end{cases}$$

Here the first case is the lift at constant velocity v. For the second case, recall that a dropped weight falls $16t^2$ ft in t sec, so the 2 sec fall must start at height $16(2^2) = 64$ ft, then drop by $16(t{-}8)^2$ ft at each time past $t = 8$.

We seek the correct speed v to make the rising and falling pieces join at $t = 8$: that is, $v \cdot 8 = s(8) = 64 - 16(8{-}8)^2$, so $v = \frac{64}{8} = 8$ ft/sec. This is continuous at $t = 8$ since: $\lim_{t\to 8^-} s(t) = \lim_{t\to 8^-} 8t = 64$, and $\lim_{t\to 8^+} s(t) = \lim_{t\to 8^+} 64 - 16(t{-}8)^2 = 64$, so the two-sided limit is $\lim_{t\to 8} s(t) = 64 = s(8)$.

Domain of continuity. Almost all functions defined by formulas are continuous, except at points where they are undefined. This follows from our methods for computing limits.

EXAMPLE: Find the points where the following function is continuous:

$$g(x) = \frac{(x^2{-}3x{+}1)\sqrt{x{+}1}}{x-3}\,.$$

First, we consider the factors outside the square root, repeatedly applying the Limit Laws from §1.6:

$$\lim_{x\to a} \frac{x^2 - 3x + 1}{x-3} \;=\; \frac{(\lim_{x\to a} x)^2 - 3\,(\lim_{x\to a} x) + 1}{(\lim_{x\to a} x) - 3} \;=\; \frac{a^2 - 3a + 1}{a - 3},$$

provided the denominator $a{-}3$ is non-zero; that is, $a \neq 3$. The Limit Laws also give $\lim_{x\to a} \sqrt{x{+}1} = \sqrt{a{+}1}$ provided $a{+}1 > 0$ to avoid the square root of a negative number; that is, for $a > -1$. Combining these, we have:

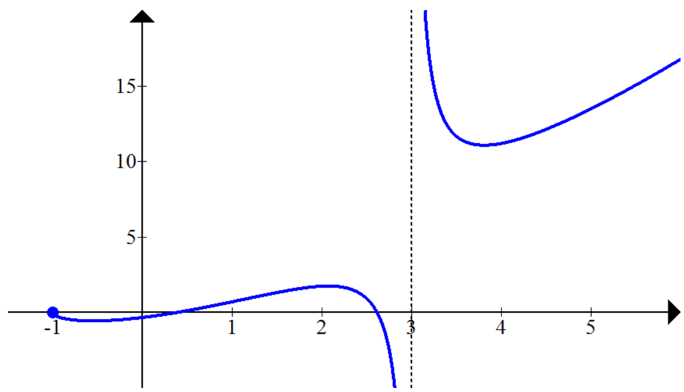
$$\lim_{x\to a} \frac{(x^2{-}3x{+}1)\sqrt{x{+}1}}{x-3} \;=\; \lim_{x\to a} \frac{x^2 - 3x + 1}{x - 3} \cdot \lim_{x\to a} \sqrt{x{+}1} \;=\; \frac{(a^2{-}3a{+}1)\sqrt{a{+}1}}{a-3}\,,$$

provided both factor limits exist, that is if $a \neq 3$ and $a > -1$. That is, $g(x)$ is continuous for all these values of a. The remaining values are:

- $a < -1$, where $g(x)$ is undefined, hence not continuous;
- $a = -1$, where $g(x)$ is continuous, since at the left endpoint of the domain of definition, we only require the one-sided limit $\lim_{x\to a^+} g(x) = g(a)$;
- $a = 3$, where the function clearly has a vertical asymptote, discontinuity of type (iv).

In summary, our $g(x)$ is continuous at every point where it is defined, that is, in the intervals[†] $[-1, 3) \cup (3, \infty)$. The graph looks like:

[†]The half-open interval $[a, b)$ is the set of all numbers x between a and b, including the left endpoint $x = a$ but excluding the right endpoint $x = b$; that is, $a \leq x < b$. The infinite interval (a, ∞) means all $x > a$, with ∞ indicating no upper bound on the right.



Composing continuous functions. Another way to combine functions $f(x)$ and $g(x)$ is to *compose* or *chain* them, taking the output of g as the input of f to obtain the new function $f(g(x))$. Composition also preserves continuity: if $g(x)$ is continuous at $x = a$, and $f(x)$ is continuous at $x = g(a)$, then $f(g(x))$ is continuous at $x = a$. This follows from the following theorem:

Composition Law: We have:

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)),$$

provided $\lim\limits_{x\to a} g(x) = b$ and $f(x)$ is continuous at $x = b$.

Proof. For any desired output error bound $\varepsilon > 0$, we must find some input accuracy $\delta > 0$ such that $|x - a| < \delta$ forces $|f(g(x)) - f(b)| < \varepsilon$.

Take any $\varepsilon > 0$. Since $f(y)$ is continuous at $y = b$, there is $\delta_1 > 0$ such that $|y - b| < \delta_1$ forces $|f(y) - f(b)| < \varepsilon$. Also, since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ forces $|g(x) - b| < \delta_1$. Therefore $0 < |x - a| < \delta$ forces $|g(x) - b| < \delta_1$, which in turn forces $|f(g(x)) - f(b)| < \varepsilon$, as required.

Intermediate Value Theorem:

> If $f(x)$ is continuous for x in the interval $[a, b]$, and r is between $f(a)$ and $f(b)$, meaning either $f(a) < r < f(b)$ or $f(a) > r > f(b)$, then there is a value[‡] $c \in (a, b)$ such that $f(c) = r$.

This says that as the function value $f(x)$ goes continuously from $f(a)$ to $f(b)$, perhaps rising and falling many times, it must pass through every value r between $f(a)$ and $f(b)$.

Note that this is *not* necessarily true for a discontinuous function like $g(x)$ in the graph above: taking $[a, b] = [2, 4]$, we have $g(2) \approx 1.7$, $g(4) \approx 11.2$, and $g(2) < 7 < g(4)$, but there is a vertical asymptote discontinuity at $t = 3$, and there is *no* $c \in (2, 4)$ with $g(c) = 7$.

However, $g(x)$ *is* continuous over the interval $[0, 1]$, with $g(0) \approx -0.33$, $g(1) \approx 0.72$, and $g(0) < 0 < g(1)$, so the Theorem says there must be some $c \in (0, 1)$ with $g(c) = 0$. This is just the x-intercept visible in the graph.

EXAMPLE: Show that there exists a solution $x = c$ to the equation $\cos(x) = x$. We have no easy way of solving this equation, but writing $f(x) = \cos(x) - x$, we know that $f(0) = 1$, $f(\pi) = -1-\pi$, and $f(0) > 0 > f(\pi)$. Since $f(x)$ is continuous, the Theorem guarantees some $c \in (0, \pi)$ with $f(c) = 0$, meaning $\cos(c) = c$.

[‡] Here c lies in the open interval (a, b), between a and b but excluding both endpoints: $a < c < b$.