Limit Laws

Operations on limits. Some general combination rules make most limit computations routine. Suppose we know that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then we have the Limit Laws:

- *Sum:* $\lim_{x\to a} (f(x) + g(x)) \;=\; \lim_{x\to a} f(x) \,+\, \lim_{x\to a} g(x)$.
- *Difference:* $\lim_{x\to a} (f(x) - g(x)) \;=\; \lim_{x\to a} f(x) \,-\, \lim_{x\to a} g(x)$.
- *Constant Multiple:* $\lim_{x\to a} (c\, f(x)) \;=\; c\, \lim_{x\to a} f(x)$, for a constant c.
- *Product:* $\lim_{x\to a} f(x)g(x) \;=\; \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
- *Quotient:* $\lim_{x\to a} \frac{f(x)}{g(x)} \;=\; \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
- *Power:* $\lim_{x\to a} f(x)^n \;=\; (\lim_{x\to a} f(x))^n$, for a whole number n.
- *Root:* $\lim_{x\to a} \sqrt[n]{f(x)} \;=\; \sqrt[n]{\lim_{x\to a} f(x)}$, for a whole number* n.

These all have the form: "The limit of an operation equals the operation applied to the limits." These Laws are also valid for one-sided limits.

Limits by plugging in. Assuming the Limit Laws and the Basic Limits $\lim_{x\to a} x = a$ and $\lim_{x\to a} c = c$, we can prove that most functions are continuous, meaning the $\lim_{x\to a} f(x)$ is obtained by substituting $x = a$ to get $f(a)$. For example, we can formally compute the limit:

$$
\begin{aligned}
\lim_{x\to 2} \frac{1-\sqrt{x}}{1+x} &= \frac{\lim\limits_{x\to 2} 1-\sqrt{x}}{\lim\limits_{x\to 2} 1+x} && \text{by the Quotient Law}^{\dagger} \\
&= \frac{\lim\limits_{x\to 2} 1 - \lim\limits_{x\to 2} \sqrt{x}}{\lim\limits_{x\to 2} 1 + \lim\limits_{x\to 2} x} && \text{by the Sum and Difference Laws} \\
&= \frac{\lim\limits_{x\to 2} 1 - \sqrt{\lim\limits_{x\to 2} x}}{\lim\limits_{x\to 2} 1 + \lim\limits_{x\to 2} x} && \text{by the Root Law} \\
&= \frac{1-\sqrt{2}}{1+2} = \frac{1-\sqrt{2}}{3} && \text{by the Basic Limits.}
\end{aligned}
$$

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*If n is even, we assume $\lim_{x\to a} f(x) > 0$.

†The Quotient Law requires that the denominator have a non-zero limit. We tentatively proceed with the computation and find the denominator to be 3, which retrospectively justifies the quotient step.

That is, the correct limit would be obtained just by substituting $x = 2$. In general, substituting $x = a$ gives the correct limit unless it leads to a meaningless expression like $\frac{0}{0}$ or $\sqrt{-1}$ (we do not consider imaginary numbers in this course). In Notes §2.4, we will show that trigonometric functions like $\sin(x)$ and $\tan(x)$ are also continuous when defined, and the same for functions like 2^x and $\log(x)$, so this principle works for pretty much all formulas.

Limits by canceling zeroes. As we have seen, the most important limits are those for which substitution gives the meaningless expression $\frac{0}{0}$. To compute these, we must cancel vanishing factors from the top and bottom, until we get an expression which can be evaluated by the Laws. This often requires factoring, for example:

$$\lim_{x\to 2} \frac{x^2 - 4x + 4}{x^2 - x - 2} \;=\; \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+1)} \;=\; \lim_{x\to 2} \frac{x-2}{x+1}\,,$$

which can be evaluated by substituting $x = 2$. Another trick to avoid $\frac{0}{0}$ is to multiply by a conjugate radical to eliminate square roots:

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} \;=\; \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} \;=\; \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x})^2 - 3^2}$$

$$=\; \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} \;=\; \lim_{x\to 9} \sqrt{x}+3 \;=\; \sqrt{9}+3 \;=\; 6.$$

Limits by cases. A familiar function defined by cases is the absolute value: $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$. To evaluate limits involving such functions, we must consider these cases separately. For example, compute:

$$\lim_{x\to 2} \frac{|x^2-4|}{x-2}\,.$$

The function is not continuous: plugging in $x = 2$ gives $\frac{0}{0}$. Rather, we must examine the cases where $x < 2$ and $x > 2$. In the first case, we deduce:[†]

$$x < 2 \;\implies\; x^2 < 4 \implies\; x^2-4 < 0 \;\implies\; |x^2-4| = -(x^2-4).$$

Thus, we have the left limit:

$$\lim_{x\to 2^-} \frac{|x^2-4|}{x-2} \;=\; \lim_{x\to 2^-} \frac{-(x^2-4)}{x-2} \;=\; \lim_{x\to 2^-} \frac{-(x-2)(x+2)}{x-2} \;=\; \lim_{x\to 2^-} -(x+2) \;=\; -4.$$

We can check this by plugging in values like $x = 1.9, 1.99, \ldots$, getting $\frac{|x^2-4|}{x-2} = -3.9,\ -3.99, \ldots \to -4$.

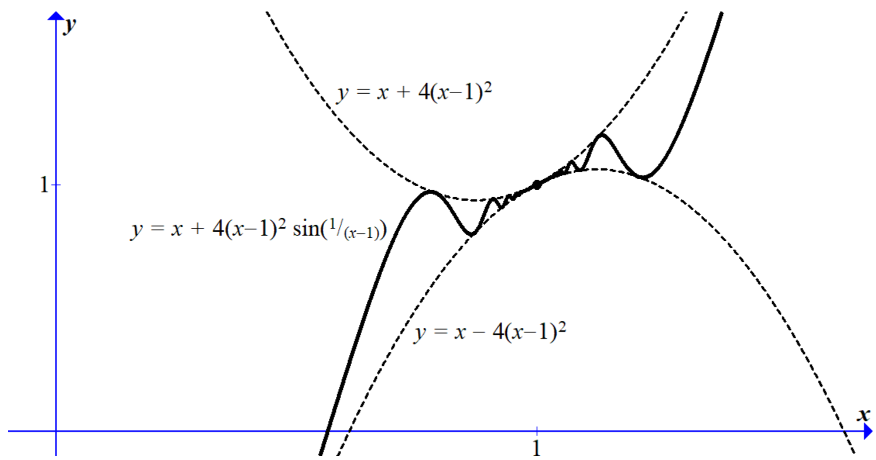
Reasoning similarly, we find $\lim_{x\to 2^+} \frac{|x^2-4|}{x-2} = \lim_{x\to 2^+}(x+2) = 4$. Since the one-sided limits disagree, the two-sided limit does not exist.

[†]The symbol $A \implies B$ means "statement A logically implies statement B".

Limits by squeezing. Some limits $\lim_{x\to a} f(x)$ are difficult to evaluate because $f(x)$ behaves erratically near $x = a$. For example:

$$\lim_{x\to 1} x + 4(x{-}1)^2 \sin(\tfrac{1}{x-1}).$$

The graph shows the weirdness, oscillating faster and faster near $x = 1$ because $\sin(\theta)$ goes through infinitely many periods as $\theta = \frac{1}{x-1}$ becomes larger.



But however complicated this behavior, we know $-1 \le \sin(\theta) \le 1$ for any θ, so our function has simple lower and upper bounds (floor and ceiling):

$$x - 4(x{-}1)^2 \;\le\; x + 4(x{-}1)^2 \sin(\tfrac{1}{x-1}) \;\le\; x + 4(x{-}1)^2.$$

Since our function lies between these bounds, so does its limit, if it exists:

$$1 \;=\; \lim_{x\to 1} x - 4(x{-}1)^2 \;\le\; \lim_{x\to 1} x + 4(x{-}1)^2 \sin(\tfrac{1}{x-1}) \;\le\; \lim_{x\to 1} x + 4(x{-}1)^2 = 1.$$

But the floor and ceiling both approach the limit $L = 1$, so our function is squeezed toward this same value:

$$\lim_{x\to 1} x + 4(x{-}1)^2 \sin(\tfrac{1}{x-1}) \;=\; 1.$$

This reasoning is formalized in the following theorem.

> *Squeeze Theorem:* Suppose $g(x) \le f(x) \le h(x)$ for all x near a (except possibly $x = a$), and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} f(x) = L$.