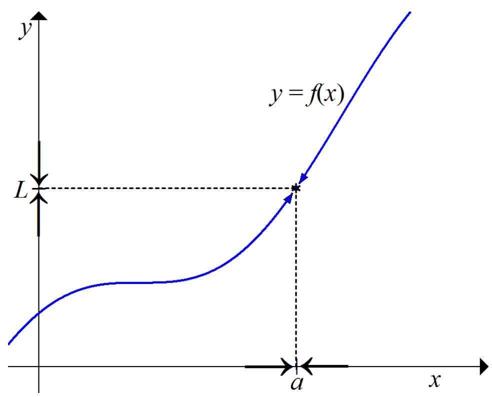
Limits

Definition of limits. The key technical tool in the previous section was the idea of a limiting value approached by approximations. We need limits for all the definitions of calculus, so we must understand them clearly.

> *Preliminary definition:* Consider a function $f(x)$ and numbers L, a. Then the *limit* of $f(x)$ equals L as x approaches a, in symbols $\lim_{x\to a} f(x) = L$, whenever $f(x)$ can be forced arbitrarily close to L by making x sufficiently close to (but unequal to) a.

That is, $f(x)$ approximates L to within any desired error tolerance, for all values of x within some small distance from a (but $x \neq a$). One more way to say it: if we make a table of $f(x)$ for any sample values of x getting closer and closer to a (such as $x = a+0.1$, $a+0.01$, etc.), then the values of $f(x)$ will get as close as we like to L (though they might never reach L). Graphically:



Evaluating limits. Some limits are easy because we can plug in $x = a$ to get the limiting value $\lim_{x\to a} f(x) = f(a)$, in which case we say $f(x)$ is *continuous* at $x = a$. Graphically, as in the above picture, this means the curve has no jump or hole at $(a, f(a))$. For example,

$$\lim_{x\to 5} x^2 \ = \ 5^2 \ = \ 25,$$

as we could see from the graph of $y = x^2$. Algebraically, if x is close enough to 5, say $x = 5 + h$ for some small h, then

$$x^2 \ = \ (5+h)^2 \ = \ 5^2 + 2(5h) + h^2 \ = \ 25 + 10h + h^2,$$

which is forced as close as we like to $L = 25$ if h is small enough (positive or negative).

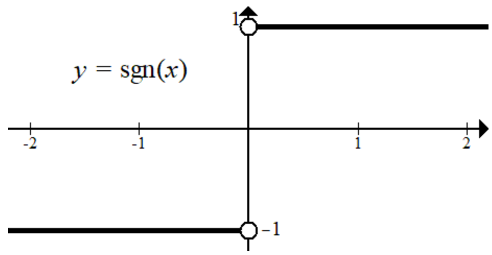
Sometimes $f(x)$ does not approach any limiting value at $x = a$, in which case we say the limit *does not exist*, and the symbol $\lim_{x\to a} f(x)$ has no

Notes by Peter Magyar magyar@math.msu.edu

meaning. For example, define the signum function $\text{sgn}(x)$ as:

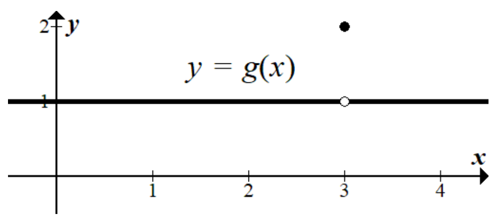
$$\text{sgn}(x) \;=\; \frac{x}{|x|} \;=\; \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \textit{undefined} & \text{for } x = 0, \end{cases}$$

with graph:



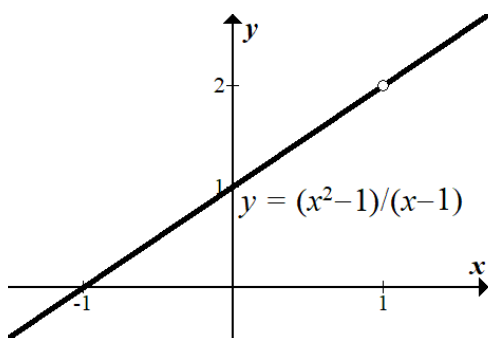
Near $x = 0$, the function cannot be forced close to any single output value. That is, $\lim_{x\to 0} \text{sgn}(x) \neq 1$, since no matter how close x gets to 0, there are some x (namely negative) for which $\text{sgn}(x)$ is far from 1; and similarly $\lim_{x\to 0} \text{sgn}(x)$ is not -1, nor 0, nor any other value. In particular, it is *false* that $\lim_{x\to 0} \text{sgn}(x) = \text{sgn}(0)$, and the function is not continuous at $x = 0$.

An important feature of $\lim_{x\to a} f(x)$ is that it does not depend on $f(a)$, even if $f(a)$ is undefined: the limit only notices values of $f(x)$ for $x \neq a$. For example, define $g(x) = 1$ for $x \neq 3$, and $g(3) = 2$, having the graph:



Then $\lim_{x\to 3} g(x) = 1$, since if x is close enough to (but unequal to) 3, then $g(x)$ is arbitrarily close to $L = 1$ (in fact $g(x) = L$). Again, $\lim_{x\to 3} g(x) \neq g(3) = 2$, and $g(x)$ is not continuous at $x = 3$.

The important limits in calculus, such as instantaneous velocity, are cases where the function is not defined at $x = a$. For example, consider $\lim_{x\to 1} \frac{x^2-1}{x-1}$. Plugging in $x = 1$ gives the meaningless expression $\frac{0}{0}$, so this function is not continuous, but the limit still exists. Indeed, plotting points gives the graph:



It seems the limit is $L = 2$: the graph approaches $(1, 2)$, so if x is sufficiently close to (but not equal to) 1, then $f(x)$ is forced as close as desired to 2. We can prove this algebraically:

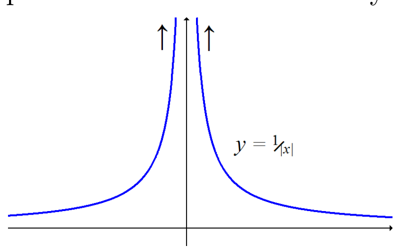
$$\lim_{x\to 1} \frac{x^2-1}{x-1} \;=\; \lim_{x\to 1} \frac{(x{-}1)(x{+}1)}{x{-}1} \;=\; \lim_{x\to 1} x{+}1 \;=\; 1+1 \;=\; 2,$$

since $x{+}1$ is continuous.

One-sided and infinite limits. We define another type of limit. *One-sided limits* (from the right or left) notice only values of x on one side of a. That is, the limit of $f(x)$ equals L as x approaches a from the right, denoted $\lim_{x\to a^+} f(x) = L$, whenever $f(x)$ can be forced arbitrarily close to L by making x sufficiently close to (but *greater* than) a. The limit from the left, denoted $\lim_{x\to a^-} f(x) = L$, is the same, except with x *less* than a.

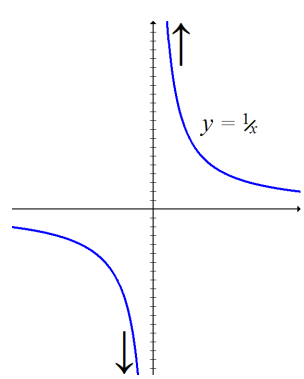
If we have the ordinary limit $\lim_{x\to a} f(x) = L$, then clearly the left and right limits have the same value L. Thus, in the above examples, we have $\lim_{x\to 5^+} x^2 = \lim_{x\to 5^-} x^2 = 5^2$, and $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x) = 0$, and $\lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2$. However, $\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$ and $\lim_{x\to 0^-} \operatorname{sgn}(x) = -1$, even though $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist.

Finally, we define *infinite limits*: $\lim_{x\to a} f(x) = \infty$ means that $f(x)$ can be forced larger than any bound (for instance $f(x) > 1000$) by making x sufficiently close to (but not equal to) a. The symbol ∞ has no meaning by itself: this is just a way of saying that $f(x)$ becomes as large a number as we like. For example, we have $\lim_{x\to 0} \frac{1}{|x|} = \infty$, since $\frac{1}{\text{tiny}} = \text{huge}$, so the graph $y = \frac{1}{|x|}$ shoots upward toward the vertical asymptote $x = 0$.



However, for the function $\frac{1}{x}$, we have $\lim_{x\to 0} \frac{1}{x} \neq \infty$, since no matter how close x is to 0, we cannot force $\frac{1}{x}$ above a given positive bound: rather, for x a tiny negative number, $\frac{1}{x} = \frac{1}{-\text{tiny}} = -\text{huge}$, a large *negative* number. In fact, the graph shoots upward to the right of the vertical asymptote, and downward to the left of the asymptote, so we have one-sided infinite limits:

$$\lim_{x\to 0^-} \frac{1}{x} \;=\; -\infty$$



$$\lim_{x\to 0^+} \frac{1}{x} \;=\; \infty$$

Vertical asymptotes. We determine the asymptotic behavior of:

$$f(x) \;=\; \frac{2x-4}{x^2-4x+3} \;=\; \frac{2(x-2)}{(x-1)(x-3)}\,.$$

Given the first form of the function, we immediately factor to see the vanishing of the numerator at $x = 2$ and the denominator at $x = 1, 3$. (These x-values are different, so no factors cancel.) The vanishing of the numerator shows when $f(x) = 0$, namely at the x-intercept $x = 2$.

The vanishing of the denominator shows when $f(x)$ becomes huge, namely near the vertical asymptotes $x = 1$ and $x = 3$. To see whether the function goes up or down near the asymptotes, we keep track of the signs.

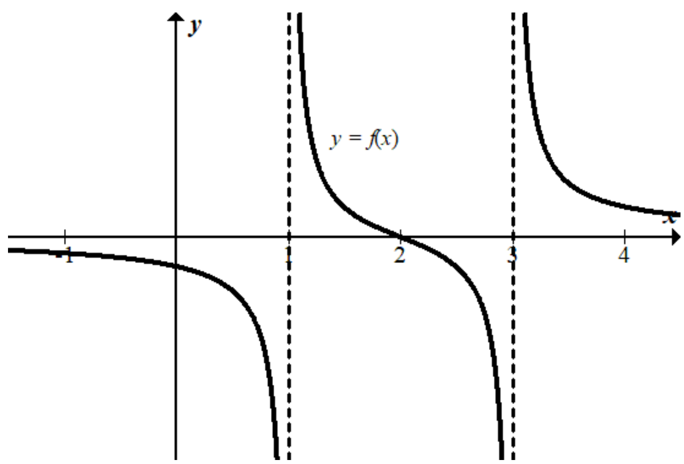
For $x < 1$, we have $x-1$, $x-2$, $x-3 < 0$ all negative:

$$f(x) \;=\; \frac{2(x-2)}{(x-1)(x-3)} \;=\; \frac{2(-)}{(-)(-)} \;=\; (-) \quad \text{so} \quad \lim_{x\to 1^-} f(x) = -\infty.$$

For $1 < x < 2$, we have $x-1 > 0$ and $x-2$, $x-3 < 0$:

$$f(x) \;=\; \frac{2(x-2)}{(x-1)(x-3)} \;=\; \frac{2(-)}{(+)(-)} \;=\; (+) \quad \text{so} \quad \lim_{x\to 1^+} f(x) = \infty.$$

Similarly, $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$. The graph is:



Of course, as for all limits we can approximate by plugging in sample inputs: for example, $f(.9) \approx -10.5$, $f(.99) \approx -100.5$, so it seems $\lim_{x\to 1^-} f(x) = -\infty$.

NOTE: In the slightly different function $\frac{2x-2}{x^2-4x+3} = \frac{2(x-1)}{(x-1)(x-3)}$, both numerator and denominator vanish at $x = 1$. The $(x-1)$ factors cancel, and the function has neither an asymptote nor an intercept at $x = 1$, only a hole in the graph where $f(1) = \frac{0}{0}$ is undefined.