Definition of limits. The key technical tool in the previous section was the idea of a limiting value approached by approximations. We need limits for all the definitions of calculus, so we must understand them clearly.

Preliminary definition: Consider a function $f(x)$ and numbers $L$, $a$. Then the limit of $f(x)$ equals $L$ as $x$ approaches $a$, in symbols $\lim_{x \to a} f(x) = L$, whenever $f(x)$ can be forced arbitrarily close to $L$ by making $x$ sufficiently close to (but unequal to) $a$.

That is, $f(x)$ approximates $L$ to within any desired error tolerance, for all values of $x$ within some small distance from $a$ (but $x \neq a$). One more way to say it: if we make a table of $f(x)$ for any sample values of $x$ getting closer and closer to $a$ (such as $x = a + 0.1$, $a + 0.01$, etc.), then the values of $f(x)$ will get as close as we like to $L$ (though they might never reach $L$). Graphically:

![Graph showing a function approaching a limit](image)

Evaluating limits. Some limits are easy because we can plug in $x = a$ to get the limiting value $\lim_{x \to a} f(x) = f(a)$, in which case we say $f(x)$ is continuous at $x = a$. Graphically, as in the above picture, this means the curve has no jump or hole at $(a, f(a))$. For example,

$$\lim_{x \to 5} x^2 = 5^2 = 25,$$

as we could see from the graph of $y = x^2$. Algebraically, if $x$ is close enough to 5, say $x = 5 + h$ for some small $h$, then

$$x^2 = (5+h)^2 = 5^2 + 2(5h) + h^2 = 25 + 10h + h^2,$$

which is forced as close as we like to $L = 25$ if $h$ is small enough (positive or negative).

Sometimes $f(x)$ does not approach any limiting value at $x = a$, in which case we say the limit does not exist, and the symbol $\lim_{x \to a} f(x)$ has no
meaning. For example, define the signum function \( \text{sgn}(x) \) as:

\[
\text{sgn}(x) = \frac{x}{|x|} = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \text{undefined} & \text{for } x = 0, \end{cases}
\]

with graph:

Near \( x = 0 \), the function cannot be forced close to any single output value. That is, \( \lim_{x \to 0} \text{sgn}(x) \neq 1 \), since no matter how close \( x \) gets to 0, there are some \( x \) (namely negative) for which \( \text{sgn}(x) \) is far from 1; and similarly \( \lim_{x \to 0} \text{sgn}(x) \) is not \(-1\), nor 0, nor any other value. In particular, it is false that \( \lim_{x \to 0} \text{sgn}(x) = \text{sgn}(0) \), and the function is not continuous at \( x = 0 \).

An important feature of \( \lim_{x \to a} f(x) \) is that it does not depend on \( f(a) \), even if \( f(a) \) is undefined: the limit only notices values of \( f(x) \) for \( x \neq a \).

For example, define \( g(x) = 1 \) for \( x \neq 3 \), and \( g(3) = 2 \), having the graph:

Then \( \lim_{x \to 3} g(x) = 1 \), since if \( x \) is close enough to (but unequal to) 3, then \( g(x) \) is arbitrarily close to \( L = 1 \) (in fact \( g(x) = L \)). Again, \( \lim_{x \to 3} g(x) \neq g(3) = 2 \), and \( g(x) \) is not continuous at \( x = 3 \).

The important limits in calculus, such as instantaneous velocity, are cases where the function is not defined at \( x = a \). For example, consider \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \). Plugging in \( x = 1 \) gives the meaningless expression \( \frac{0}{0} \), so this function is not continuous, but the limit still exists. Indeed, plotting points gives the graph:

It seems the limit is \( L = 2 \): the graph approaches \((1, 2)\), so if \( x \) is sufficiently close to (but not equal to) 1, then \( f(x) \) is forced as close as desired to 2. We
can prove this algebraically:

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} x+1 = 1 + 1 = 2,
\]

since \(x+1\) is continuous.

**One-sided and infinite limits.** We define another type of limit. *One-sided limits* (from the right or left) notice only values of \(x\) on one side of \(a\). That is, the limit of \(f(x)\) equals \(L\) as \(x\) approaches \(a\) from the right, denoted \(\lim_{x \to a^+} f(x) = L\), whenever \(f(x)\) can be forced arbitrarily close to \(L\) by making \(x\) sufficiently close to (but greater than) \(a\). The limit from the left, denoted \(\lim_{x \to a^-} f(x) = L\), is the same, except with \(x\) less than \(a\).

If we have the ordinary limit \(\lim_{x \to a} f(x) = L\), then clearly the left and right limits have the same value \(L\). Thus, in the above examples, we have \(\lim_{x \to 5^+} x^2 = \lim_{x \to 5^-} x^2 = 5^2\), and \(\lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} g(x) = 0\), and \(\lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = 2\). However, \(\lim_{x \to 0^+} \text{sgn}(x) = 1\) and \(\lim_{x \to 0^-} \text{sgn}(x) = -1\), even though \(\lim_{x \to 0} \text{sgn}(x)\) does not exist.

Finally, we define *infinite limits*: \(\lim_{x \to a} f(x) = \infty\) means that \(f(x)\) can be forced larger than any bound (for instance \(f(x) > 1000\)) by making \(x\) sufficiently close to (but not equal to) \(a\). The symbol \(\infty\) has no meaning by itself: this is just a way of saying that \(f(x)\) becomes as large a number as we like. For example, we have \(\lim_{x \to 0} \frac{1}{|x|} = \infty\), since \(\frac{1}{\text{tiny}} = \text{huge}\), so the graph \(y = \frac{1}{|x|}\) shoots upward toward the vertical asymptote \(x = 0\).

However, for the function \(\frac{1}{x}\), we have \(\lim_{x \to 0} \frac{1}{x} \neq \infty\), since no matter how close \(x\) is to 0, we cannot force \(\frac{1}{x}\) above a given positive bound: rather, for \(x\) a tiny negative number, \(\frac{1}{x} = \frac{1}{-\text{tiny}} = -\text{huge}\), a large *negative* number. In fact, the graph shoots upward to the right of the vertical asymptote, and downward to the left of the asymptote, so we have one-sided infinite limits:
**Vertical asymptotes.** We determine the asymptotic behavior of:

\[ f(x) = \frac{2x - 4}{x^2 - 4x + 3} = \frac{2(x-2)}{(x-1)(x-3)}. \]

Given the first form of the function, we immediately factor to see the vanishing of the numerator at \( x = 2 \) and the denominator at \( x = 1, 3 \). (These \( x \)-values are different, so no factors cancel.) The vanishing of the numerator shows when \( f(x) = 0 \), namely at the \( x \)-intercept \( x = 2 \).

The vanishing of the denominator shows when \( f(x) \) becomes huge, namely near the vertical asymptotes \( x = 1 \) and \( x = 3 \). To see whether the function goes up or down near the asymptotes, we keep track of the signs.

For \( x < 1 \), we have \( x-1, x-2, x-3 < 0 \) all negative:

\[ f(x) = \frac{2(x-2)}{(x-1)(x-3)} = \frac{2(-)}{(-)(-)} = (-) \text{ so } \lim_{x \to 1^-} f(x) = -\infty. \]

For \( 1 < x < 2 \), we have \( x-1 > 0 \) and \( x-2, x-3 < 0 \):

\[ f(x) = \frac{2(x-2)}{(x-1)(x-3)} = \frac{2(-)}{(+)(-)} = (+) \text{ so } \lim_{x \to 1^+} f(x) = \infty. \]

Similarly, \( \lim_{x \to 3^-} f(x) = -\infty \) and \( \lim_{x \to 3^+} f(x) = \infty \). The graph is:

![Graph of the function showing vertical asymptotes at x=1 and x=3.]

Of course, as for all limits we can approximate by plugging in sample inputs: for example, \( f(.9) \approx -10.5 \), \( f(.99) \approx -100.5 \), so it seems \( \lim_{x \to 1^-} f(x) = -\infty \).

**NOTE:** In the slightly different function \( \frac{2x-2}{x^2-4x+3} = \frac{2(x-1)}{(x-1)(x-3)} \), both numerator and denominator vanish at \( x = 1 \). The \( (x-1) \) factors cancel, and the function has neither an asymptote nor an intercept at \( x = 1 \), only a hole in the graph where \( f(1) = \frac{0}{0} \) is undefined.