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Math 481 **Quiz 27** **Mar 29, 2024**

PROPOSITION: Let G be a connected graph containing a cycle C with edge $e = xy$, so $e \subset C \subset G$. Then the graph $G - e$, removing e but keeping vertices x, y, is connected.

Sketch a proof, first writing the hypothesis at the top and the conclusion at the bottom, with their definitions. Notation: let $C = xeyQx$, from x along e to y, then along path Q back to x.

Informal argument: We need to show any two vertices v, w are connected in $G - e$. There is a path vPw in G, which might contain the edge $e = xy$ in C. In that case, avoid e by going the other way around C, getting a new path which connects v to w in $G - e$.

Formal proof: By hypothesis, assume that G is a connected graph, meaning any vertices v, w are connected by a path $vPw \subset G$. Also, G contains a cycle of the form $C = xeyQx$, where $e = xy$ is an edge and yQx is a path not containing e. Let $G{-}e$ be the graph with the same vertices as G but missing edge e.

Since G is connected, any two vertices v, w are connected by some path vPw in G. If P does not contain e, then $P' = P$ is a path $vP'w \subset G{-}e$.

If P does contain $e = xy$, we may write it as vP_1xeyP_2w, and construct

$$R = vP_1xQyP_2w,$$

the walk from v along P to x, then around Q to y, continuing along P to w. By definition, the path P and the cycle C can use e only once, so the path segments vPx, xQy, yPw cannot contain e. Thus R is a walk in $G{-}e$ connecting v, w, and we can cut it down to a path $vP'w \subset G{-}e$.

Either way, arbitrary vertices v, w are connected by $vP'w$ in $G{-}e$, so it is a connected graph.