NAME:__________________________

Math 481 **Quiz 23 Solution** **Mar 20, 2024**

1. Define the distance between vertices in a connected graph: $\text{dist}(v, w)$ means ...?

Solution: For vertices v, w in a graph G, the distance $\text{dist}(v, w)$ is the minimum number of edges in a path from v to w inside G, i.e. the length of a shortest path $vPw \subset G$.

Note: We could also say "minumum number of edges in a *walk*", since the shortest walk is always a path (no repeat vertices).

2. TRUE OR FALSE: For any distinct vertices x, y, z in a connected graph,

$$\text{dist}(x, z) = \text{dist}(x, y) + \text{dist}(y, z).$$

Solution: False. A counterexample is the complete graph K_3 with vertices $V = \{x, y, z\}$, having $\text{dist}(x, y) = \text{dist}(x, z) = \text{dist}(y, z) = 1$, but $1 \neq 1 + 1$.

3. TRUE OR FALSE: For any distinct vertices x, y, z in a connected graph,

$$\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z).$$

Solution: True. Informally, a shortest path from x to y joined to a shortest path from y to z is a walk from x to z with $\text{dist}(x, y) + \text{dist}(y, z)$ edges, and the shortest xz-walk (or path), having $\text{dist}(x, z)$ edges, cannot be longer than this.

Formal proof: Let xPy and yQz be shortest paths. Then their concatenation $xPyQz$ is a walk from x to z of length $\text{dist}(x, y) + \text{dist}(y, z)$. This is not necessarily a path, since Q might cross or backtrack P; however, we know that any walk can be cut down to a path with the same endpoints by skipping any loops. Thus there exists a path from x to z of length at most $\text{dist}(x, y) + \text{dist}(y, z)$, and a shortest path of length $\text{dist}(x, z)$ cannot be longer than this.

Note: This remains true for a disconnected graph, in which some distances can be infinte:
if $\text{dist}(x, z) = \infty$, then $\text{dist}(x, y)$ or $\text{dist}(y, z)$ must be ∞.