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Math 481 **Quiz 13 Solution** **Feb 9, 2024**

Consider the sequence defined by the recurrence:

$$a_0 = 0, \qquad a_k = 3a_{k-1} + 1 \text{ for } k \geq 1.$$

That is, $a_0 = 0$, $a_1 = 3(0)+1 = 1$, $a_2 = 3(1)+1 = 4$, $a_3 = 3(4)+1 = 13$, etc.

PROBLEM: Solve this recurrence by the Method of Generating Functions.

Step 1: Substitute the recurrence into the generating function, to write $f(x)$ as an expression involving $f(x)$. Then solve the resulting equation for $f(x)$.

$$\begin{aligned} f(x) &= a_0 + \sum_{k\geq 1} a_k x^k = 0 + \sum_{k\geq 1}(3a_{k-1}+1)x^k \\ &= 3x\sum_{k\geq 1} a_{k-1}x^{k-1} + x\sum_{k\geq 1} x^{k-1} \\ f(x) &= 3xf(x) + \frac{x}{1-x}. \end{aligned}$$

We used the change of index $f(x) = \sum_{k\geq 0} a_k x^k = \sum_{k\geq 1} a_{k-1}x^{k-1} = a_0 + a_1 x + a_2 x^2 + \cdots$. Solving for $f(x)$ gives:

$$f(x)(1-3x) = \frac{x}{1-x}, \qquad f(x) = \frac{x}{(1-3x)(1-x)}.$$

Step 2: Expand $f(x)$ into partial fractions, and give an explicit formula for a_k.

$$f(x) = \frac{x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

For all x: $\quad x = A(1-x) + B(1-3x)$

For $x = 1$: $\quad 1 = A(1-1) + B(1-3) \quad \Longrightarrow \quad B = -\frac{1}{2}$

For $x = \frac{1}{3}$: $\quad \frac{1}{3} = A(1-\frac{1}{3}) + B(1-3(\frac{1}{3})) \quad \Longrightarrow \quad A = \frac{1}{2}$

$$f(x) = \frac{\frac{1}{2}}{1-3x} + \frac{\frac{1}{2}}{1-x} = \frac{1}{2}\sum_{k\geq 0}(3x)^k - \frac{1}{2}\sum_{k\geq 0} x^k = \sum_{k\geq 0}(\tfrac{1}{2}3^k - \tfrac{1}{2})x^k.$$

The coefficient of x^k is thus:

$$a_k = \frac{3^k - 1}{2} \text{ for all } k \geq 0.$$

For example $a_0 = \frac{1}{2}(3^0 - 1) = 0$ and $a_3 = \frac{1}{2}(3^3 - 1) = \frac{1}{2}(26) = 13$.