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**Math 481** **Quiz 12 Solution** **Feb 7, 2024**

QUESTION: A menu offers \$1 Fries, \$2 Hamburger, \$2 Shake. How many ways to spend \$10?

EXAMPLE: Buy 4F, 1H, 2S, for a total of $4(\$1) + 1(\$2) + 2(\$2) = \$10$.

Solve this using the Method of Generating Functions, as follows.

*Step 0:* Generalize to a family of problems $a_0, a_1, a_2, \dots$

*Solution:* $a_k$ counts the number of ways to spend $\$k$.

*Step 1:* Write a logical algorithm to produce the objects any size, and translate this into a product formula for the generating function $f(x) = \sum_{k\geq 0} a_k x^k$. *Solution:*

$$\begin{pmatrix}\text{buy any}\\ \text{amount}\end{pmatrix} \iff (0F \text{ or } 1F \text{ or } 2F \text{ or } \cdots) \,\&\, (0H \text{ or } 1H \text{ or } 2H \text{ or } \cdots) \,\&\, (0S \text{ or } 1S \text{ or } 2S \text{ or } \cdots)$$

Using the logic/algebra dictionary, each choice of items costing $\$m$ is replaced with $x^m$:

$$\begin{aligned} f(x) &= (1+x+x^2+\cdots)\cdot(1+x^2+x^4+\cdots)\cdot(1+x^2+x^4+\cdots) \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)(1-x^2)^2}. \end{aligned}$$

*Step 2:* Use known series to find the Taylor series of $f(x)$. Give an explicit formula for $a_k$. *Hints:* $\frac{1}{1-x} = \frac{(1+x)}{(1-x)(1+x)} = \frac{1+x}{1-x^2}$. Also $\frac{1}{(1-z)^n} = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) z^k$, where $z$ is any quantity.

*Solution:* It is difficult to expand $f(x)$ as a product of known series. We would like to reduce the denominator to be a power of a single binomial $(1-z)^n$.

$$\begin{aligned} f(x) &= \frac{1}{(1-x)}\,\frac{1}{(1-x^2)^2} = \frac{(1+x)}{(1+x)(1-x)}\,\frac{1}{(1-x^2)^2} \\ &= \frac{1+x}{(1-x^2)}\,\frac{1}{(1-x^2)^2} = (1+x)\,\frac{1}{(1-x^2)^3} \\ &= (1+x)\sum_{k\geq 0}\left(\!\binom{3}{k}\!\right)(x^2)^k = \sum_{k\geq 0}\left(\!\binom{3}{k}\!\right)x^{2k} + \left(\!\binom{3}{k}\!\right)x^{2k+1} \end{aligned}$$

Thus the coefficients of the even terms $x^{2k}$ and the odd terms $x^{2k+1}$ are:

$$a_{2k} = a_{2k+1} = \left(\!\binom{3}{k}\!\right) = \binom{k+2}{k} = \binom{k+2}{2} = \tfrac{1}{2}(k+1)(k+2).$$

In particular, for $k=5$ we have $a_{10} = \frac{1}{2}(6)(7) = 21$, so there are 21 ways to spend \$10.