**Math 481** # Midterm Review Problems **Oct 2021**

**1.** There are 100 runners in the state athletic division. Count how many possible results in each case below.

**a.** They all run a race, and we rank all runners in order of finishing. *Ans:* 100!

**b.** In the race, we rank only the top 10 runners in order, ignoring the rest. *Ans:* $100^{\underline{10}}$

**c.** We choose a State Team of 10 runners. *Ans:* $\binom{100}{10}$

**d.** Choose a State Team of 10 runners, including captain and vice-captain. *Ans:* $\binom{100}{10} 10^{\underline{2}}$

**e.** We distribute 10 named awards (A award, B award,...) among the 100 runners, where each runner can receive multiple awards. *Ans:* $100^{10}$. *Choose who gets each award.*

**f.** We distribute 10 named awards among the 100 runners, where each runner can receive at most one award. *Ans:* $100^{\underline{10}}$. *Equivalent to (b).*

**2.** We have 10 books to arrange on 4 shelves. Count how many possible arrangements in each case below.

**a.** 10 identical notebooks. Example: NNNNN, NN, none, NNN. *Ans:* $\left(\!\binom{4}{10}\!\right)$

**b.** 5 blue notebooks, 5 red notebooks. Example: BRRBR, BB, none, BRR.
*Hint:* Arrange 10 blank N's, then replace 5 of them by R's, the rest by B's. *Ans:* $\left(\!\binom{4}{10}\!\right)\binom{10}{5}$

**c.** 10 distinct novels A, B, ..., J. Example: BJFCH, DA, none, EIG
*Hint:* Arrange 10 blank notebooks, then replace with a sequence of novels. *Ans:* $\left(\!\binom{4}{10}\!\right)$ 10!

**d.** 10 novels arranged alphabetically on each shelf. Example: BCFHJ, AD, none, EGI.
*Hint:* Place A on a shelf, then B, etc. How many choices for each? *Ans:* $4^{10}$

**3.** Arrange 10 novels on 4 shelves, alphabetically on each shelf, with *no empty shelves*. Example: BCFHJ, D, A, EGI. Solve this by PIE.

**a.** Define a set of all arrangements $A$, and four sets of bad arrangements $B_1, \ldots, B_4$.

**b.** Apply the PIE formula and compute the terms. *Ans:* $4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}2^{10} - \binom{4}{3}1^{10}$

**4.** Problem: There are one million numbers from 0 to 999,999. How many have digit sum equal to 27? Example: The number 67,338 has sum 6+7+3+3+8 = 27.

Solve this problem by the Method of Generating Functions.

**a.** Step 0: Generalize to a family of counting problems with answers $a_0, a_1, a_2, \ldots$, so that the original problem is $a_{27}$. That is, $a_k$ counts the number of ....

**b.** Step 1: Find a simple expression for $f(x) = \sum_{k\geq 0} a_k x^k$. Write an algorithm to select 6 digits, translate it into an expression for $f(x)$, and simplify using geometric series.

**c.** Step 2: Explicitly find the Taylor series for $f(x)$ using our knowledge of functions.
*Hint:* Use binomial and negative binomial series, collect all $x^{27}$ terms, and find $a_{27}$.
*Ans:*

$$f(x) = (x^0 + x^1 + \cdots + x^9)^6 = \left(\frac{1-x^{10}}{1-x}\right)^6 = (1-x^{10})^6 \frac{1}{(1-x)^6}$$

$$a_{27} = \binom{6}{0}\left(\!\binom{6}{27}\!\right) - \binom{6}{1}\left(\!\binom{6}{17}\!\right) + \binom{6}{2}\left(\!\binom{6}{7}\!\right)$$