PROPOSITION. The following identity holds for any whole number n:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^{2}.$$

1. Write out Pascal's Triangle with all the numbers $\binom{m}{k}$ up to m = 8. Use this to check the above identity for n = 0, 1, 2, 3, 4 by direct computation.

2. To set up a transformation proof, we must first describe each side of the equation as counting some class of combinatorial objects. By definition, the left side $\binom{2n}{n}$ is the number of subsets S of n elements inside [2n].¹

The right side is more complicated: it is the set of pairs of subsets (S_1, S_2) , both with the same number of elements, and both inside [n].

Problem: Use the Addition and Multiplication Principles to count the possible choices for (S_1, S_2) , obtaining the formula on the right side. (The summation is over all possible sizes $k = |S_1| = |S_2|$.)

3. For n = 3, write down all $\binom{6}{3} = 20$ sets $S \subset [6]$ in one column, and all the pairs (S_1, S_2) in another column. Find a natural way to transform the 3-element set S into two sets $S_1, S_2 \subset [3]$ of equal size, in a reversible way that does not lose any data. Draw lines connecting the corresponding items on the left and right.

Hint: Start by splitting $S = S_1 \cup S'_2$ into the parts lying in the lower and upper halves of $[6] = \{1, 2, 3\} \cup \{4, 5, 6\}$. However S'_2 is not in [3] and doesn't have the same size as S_1 , so you have to do more transforming to get S_2 .

4. Formally define this transformation for any n using set notation. Start with

$$[2n] = [1, n] \cup [n+1, 2n],$$

and construct S_1 and S_2 from S using set operations \cup, \cap, \setminus , and also the shift operation $S + m = \{s + m \text{ for } s \in S\}$. (Be careful to distinguish the operation $S \setminus \{m\}$ removing m, versus S - m shifting down by m.)

Reformulate the definitions of S_1, S_2 in set-builder notation, specifying conditions for the elements of each set: $S_1 = \{s \in [n] \text{ with } \dots\}, S_2 = \{s \in [n] \text{ with } \dots\}.$

5. Define the reverse transform or inverse mapping, showing how to combine S_1 and S_2 to recover S. Again use set operations, then also set-builder notation.

Extra Credit: These transforms essentially prove the identity by the Transformation Principle. Complete the proof by formally checking that the mappings $T(S) = (S_1, S_2)$ and $T'(S_1, S_2) = S$ reverse each other:

$$T'(T(S)) = S$$
 and $T(T'(S_1, S_2)) = (S_1, S_2).$

¹Notation: $[m] = \{1, 2, ..., m\}$ and $[\ell, m] = \{\ell, \ell+1, ..., m-1, m\}$.