Math 481 | **Hand-In Homework 1** | **Jan 2024**

PROPOSITION. The following identity holds for any whole number n:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

1. Write out Pascal's Triangle with all the numbers $\binom{m}{k}$ up to $m = 8$. Use this to check the above identity for $n = 0, 1, 2, 3, 4$ by direct computation.

2. To set up a transformation proof, we must describe each side of the equation as naturally counting some class of combinatorial objects. By definition, the left side $\binom{2n}{n}$ is the number of subsets S of n elements inside $[2n]$.[1]

The right side is more complicated: it is the set of pairs of subsets (S_1, S_2), both with the same number of elements, and both inside $[n]$.

Problem: Explain why this counts the right side, using the Sum and Product Principles. (The summation is over all k for which $\binom{n}{k}$ makes sense, so think of k as an arbitrary size.)

3. For $n = 3$, write down all $\binom{6}{3} = 20$ sets $S \subset [6]$ in one column, and all the pairs (S_1, S_2) in another column. Find a natural way to transform the 3-element set S into two sets $S_1, S_2 \subset [3]$ of equal size, in a reversible way that does not lose any data. Show this transformation by drawing lines connecting the items on the left and right.

Hint: Start by splitting $S = S_1 \cup S_2'$ into the parts lying in the lower and upper halves of $[6] = \{1, 2, 3\} \cup \{4, 5, 6\}$. Here S_2' is not in $[3]$ and doesn't have the same size as S_1, so you have to do more transforming to get S_2.

4. Define the above transformation for any n using set notation. Start with

$$[2n] = [1, n] \cup [n{+}1, 2n],$$

and construct S_1 and S_2 from S using set operations $\cup, \cap, \setminus$, and also the shift operation $S + m = \{s + m \text{ for } s \in S\}$. (Be careful to distinguish the operation $S \setminus \{m\}$ removing m, versus $S - m$ shifting down by m.)

Also define S_1, S_2 in terms of set-builder notation, giving precise conditions for the elements of each set: $S_1 = \{s \in [n] \text{ with } \ldots\}$, etc.

5. Define the reverse transform or inverse mapping, showing how to combine S_1, S_2 to recover S. Again use set operations, then also set-builder notation.

Note: These transforms prove the identity by the Transformation Principle.

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[1] Notation: $[m] = \{1, 2, \ldots, m\}$ and $[\ell, m] = \{\ell, \ell{+}1, \ldots, m{-}1, m\}$.