The Egg-Drop Numbers

Author(s): Michael Boardman

Source: *Mathematics Magazine*, Dec., 2004, Vol. 77, No. 5 (Dec., 2004), pp. 368-372

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: https://www.jstor.org/stable/3219201

NOTES

The Egg-Drop Numbers

MICHAEL BOARDMAN
Pacific University
Forest Grove, OR 97116
boardman@pacificu.edu

Each month, the mathematics students at Pacific University are given a challenge problem in what we call the "Pizza Problem of the Month." One of the most rewarding problems came from *Which Way Did the Bicycle Go?* [**3**, p. 53]:

> **An egg-drop experiment** *We wish to know which windows in a thirty-six-story building are safe to drop eggs from, and which are high enough to cause the eggs to break on landing.... Suppose two eggs are available. What is the* least *number of egg-droppings that is guaranteed to work in all cases?*

To make this problem mathematical, the authors make some simplifying assumptions including

- Eggs that survive can be used again and are not weakened. Eggs that break are history.
- Eggs that break at a particular floor would break from higher floors as well.
- Eggs that survive from a particular floor would survive from lower floors as well.

This problem piqued my interest. Unaware of the solution in the appendix, I found my own solution and stayed up late working out a nice presentation. The next morning, I shared this with a colleague who responded "Interesting. I wonder if this works with more eggs?" This Note answers the question. In working on it, I found that the egg-drop problem is a fruitful setting in which to introduce students to recurrence relations, generating functions, and other counting methods.

Two eggs, thirty-six floors To find a solution to the original problem, imagine that we have one egg rather than two. In this case, we must start at the first floor and work our way up one floor at a time until we have discovered where eggs begin breaking. When we have two eggs, we can use a more efficient strategy until the first egg breaks. Then, we must resort to the original one-egg strategy.

Table 1 illustrates our solution, which begins with a drop from the eighth floor. If this egg breaks, we move back down to the first floor and work our way up until the second egg breaks, unless it survives the drop from the seventh floor, in which case there is no need to drop it from the eighth. If the first egg does not break, we can move up to the fifteenth floor and repeat the process. In any case, we will have made no more than 8 drops. It is easy to see that this solution is optimal; if we started with a drop from the seventh floor, we would not be able to reach the thirty-sixth in 8 drops, whereas if we started on the ninth floor, we might need 9 drops to learn that the egg would break on the eighth.

TABLE 1: Where to drop the eggs

First Egg	Second Egg if First Egg Breaks												
8	1	→	2	→	3	→	4	→	5	→	6	→	7
15	9	→	10	→	11	→	12	→	13	→	14		
21	16	→	17	→	18	→	19	→	20				
26	22	→	23	→	24	→	25						
30	27	→	28	→	29								
33	31	→	32										
35	34												
36													

More eggs, more drops Our natural inclination is to generalize this algorithm for taller buildings. As we do so, we note a tricky feature of the original problem: It asked us to determine that 8 drops would be optimal, given 2 eggs and thirty-six floors, but it is actually simpler to determine that thirty-six floors can be reached, given 2 eggs and 8 drops.

FIGURE 1 shows which floors we can reach for various numbers of drops of 2 eggs and leads us to state the problem in a different way:

A generalized egg-drop problem *Given k eggs, how many floors can we reach if we have at most n drops?*

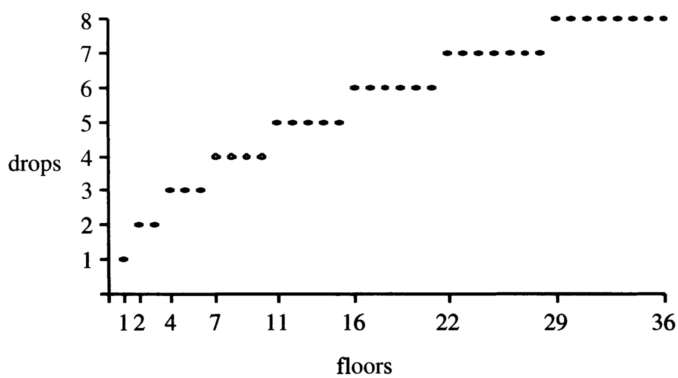

Figure 1 Solutions to two-egg problem

We label the answer to this question $\left\langle {n \atop k} \right\rangle$ (read "n drop k") and refer to the collection of all such numbers as the *egg-drop numbers*. Our goal is to characterize and compute these numbers, directly if possible.

The heart of the solution to our original problem lies in the recurrence behavior. When the first egg broke, we had only one egg left and needed to resort to the one-egg solution. We can use this idea to construct a recurrence relation for $\left\langle {n \atop k} \right\rangle$. For when we begin with k eggs and n available drops, after the first drop, we either still have k eggs (the egg survived) or we have $k-1$ eggs (the egg broke). In either case, we have $n-1$ drops available. This reasoning, as illustrated in FIGURE 2, provides our recurrence relation:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n-1 \atop k} \right\rangle + \left\langle {n-1 \atop k-1} \right\rangle + 1 \quad \text{for all } n \geq 1, k \geq 1,$$

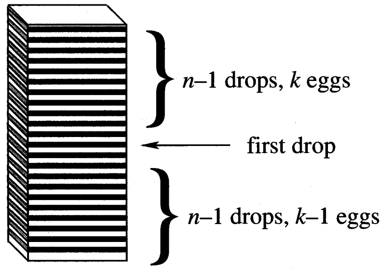

Figure 2 Motivating the recurrence relation

with

$$\left\langle {0 \atop k} \right\rangle = 0 \quad \text{for all } k \geq 0, \qquad \left\langle {n \atop 0} \right\rangle = 0 \quad \text{for all } n \geq 0.$$

The boundary conditions in the recurrence relation come from the fact that we cannot do anything without drops or eggs.

We now can recursively calculate the egg-drop numbers. Table 2 gives some of their values. You may recognize some of these numbers. For instance, the $k = 2$ column is made up of the triangular numbers (as can be seen from the solution to the two-egg problem). We would like be able to directly calculate egg-drop numbers. We illustrate two approaches to this problem.

TABLE 2: Some egg-drop numbers $\langle {n \atop k} \rangle$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	2	3	3	3	3	3	3	3
3	0	3	6	7	7	7	7	7	7
4	0	4	10	14	15	15	15	15	15
5	0	5	15	25	30	31	31	31	31
6	0	6	21	41	56	62	63	63	63
7	0	7	28	63	98	119	126	127	127
8	0	8	36	92	162	218	246	254	255

Generating Functions Given an infinite sequence $\{a_k\}_{k=0}^{\infty}$, the *ordinary power series generating function* (*ogf*) of the sequence is the symbolic power series

$$g(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$$

*ogf*s are used extensively in combinatorics to find or solve recurrence relations and to discover relations between sequences. They are especially useful because we can manipulate them symbolically, in the ring of formal power series. Thus, we do not need to concern ourselves with issues of convergence. Wilf **[2]** gives a comprehensive study of the use of *ogf*s.

Because the egg-drop numbers are doubly-indexed, with n and k, we can construct a one-parameter sequence of generating functions, $\{g_n\}$. For each $n \geq 1$,

$$g_n(x) = \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle x^k = \left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle x + \left\langle {n \atop 2} \right\rangle x^2 + \cdots + \left\langle {n \atop j} \right\rangle x^j + \cdots$$

Note that the constant term $\langle {n \atop 0} \rangle$ drops off as it is 0.

We begin by showing that our generating functions satisfy a recurrence relation related to the recurrence relation that defines the egg-drop numbers. We offer two proofs of this lemma to illustrate various ways to work with *ogf*s.

LEMMA 1. *For each* $n \geq 1$, $g_n(x) = (1+x)g_{n-1}(x) + g_1(x)$

Proof.

$$\begin{aligned} g_n(x) - g_{n-1}(x) &= \sum_{k=1}^{\infty} \left\langle {n \atop k} \right\rangle x^k - \sum_{k=1}^{\infty} \left\langle {n-1 \atop k} \right\rangle x^k = \sum_{k=1}^{\infty} \left(\left\langle {n \atop k} \right\rangle - \left\langle {n-1 \atop k} \right\rangle \right) x^k \\ &= \sum_{k=1}^{\infty} \left(\left\langle {n-1 \atop k-1} \right\rangle + 1 \right) x^k = \sum_{k=1}^{\infty} \left\langle {n-1 \atop k-1} \right\rangle x^k + \sum_{k=1}^{\infty} x^k \\ &= x \cdot g_{n-1}(x) + g_1(x) \end{aligned}$$

■

Alternate Proof. This more general method, discussed by Wilf [**2**] takes us from the recurrence relation for $\langle {n \atop k} \rangle$ to a recurrence relation for the sequence of generating functions. We multiply each term of the recurrence relation by x^k and sum over all k for which this recurrence is valid. In our case, for fixed $n \geq 1$,

$$\left\langle {n \atop k} \right\rangle = \left\langle {n-1 \atop k} \right\rangle + \left\langle {n-1 \atop k-1} \right\rangle + 1, \quad \text{for } k \geq 1.$$

Thus,

$$\sum_{k=1}^{\infty} \left\langle {n \atop k} \right\rangle x^k = \sum_{k=1}^{\infty} \left\langle {n-1 \atop k} \right\rangle x^k + \sum_{k=1}^{\infty} \left\langle {n-1 \atop k-1} \right\rangle x^k + \sum_{k=1}^{\infty} x^k$$

And hence

$$g_n(x) = g_{n-1}(x) + x g_{n-1}(x) + g_1(x) = (1+x)g_{n-1}(x) + g_1(x)$$

■

We can use this recurrence relation to find a closed form for each generating function. The proof of the following lemma, left to the reader, is a good exercise in induction.

LEMMA 2. *For each* $n \geq 1$,

$$g_n(x) = \frac{(1+x)^n - 1}{1-x}$$

Now that we know the generating functions, $g_n(x)$, we obtain our main result.

THEOREM. $\left\langle {n \atop k} \right\rangle = \sum_{j=1}^{k} \binom{n}{j}$

Proof. On the one hand,

$$g_n(x) = \left\langle {n \atop 1} \right\rangle x + \left\langle {n \atop 2} \right\rangle x^2 + \cdots + \left\langle {n \atop n} \right\rangle x^n + \left\langle {n \atop n+1} \right\rangle x^{n+1} + \cdots$$

On the other hand,

$$\begin{aligned}
g_n(x) &= \frac{1}{1-x}\cdot\left((1+x)^n - 1\right)\\
&= (1+x+x^2+x^3+\cdots)\left(\binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)\\
&= \binom{n}{1}x + \left[\binom{n}{1}+\binom{n}{2}\right]x^2 + \cdots + \left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]x^n\\
&\quad + \left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]x^{n+1}\\
&\quad + \left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]x^{n+2}+\cdots
\end{aligned}$$

Thus, g_n is also the generating function of $\sum_{j=1}^{k}\binom{n}{j}$. But two sequences with the same generating function are equal. ■

Direct counting approach Given the simplicity of the formula for the egg-drop numbers, one might suspect that there is a direct counting technique we could use. Indeed, there is. Consider a specific sequence of at most n drops with k eggs. Each of the drops has two possible outcomes: either the egg breaks or the egg does not break. Let 0 represent a drop without a break and 1 a drop with a break. Our sequence of drops thus yields a binary word of length at most n and having between zero and k 1s.

With this representation of drops, each word corresponds to a unique floor. For example, in the case of 8 drops and 2 eggs, the word 01001 corresponds to floor 11. In the general case, we make words of length $m \le n$ have length n by adding $n - m$ trailing zeros. Note that we only need do this for words that have exactly k 1s, since the efficiency of our procedure ensures that words with fewer than k broken eggs are guaranteed to use all n drops. Also note that the trailing zeros are merely placeholders and do not represent drops.

Thus, there is a one-to-one correspondence between the floors in our building and the number of binary words of length n with at least one and no more than k 1s. The latter is easily shown to be $\sum_{j=1}^{k}\binom{n}{j}$ while the former is $\left\langle {n \atop k} \right\rangle$.

Conclusion Our main theorem is both exciting and disheartening. For we have found a beautiful characterization of the egg-drop numbers. But, it is well known that there is no closed form (that is, direct formula) for the partial sum of binomial coefficients **[1]**. Alas, we cannot calculate the egg-drop numbers without a tedious recursive calculation. Nevertheless, students may find the egg-drop problem a fun way to learn the power of generating functions and other basic combinatorial tools.

Acknowledgment. Sincere thank you to the referees for their suggestions for improvement.

REFERENCES

1. M. Petkovsek, H. Wilf, and D. Zeilberger, *A = B*, A. K. Peters, Wellesley, MA, 1996.
2. H. Wilf, *Generatingfunctionology*, Academic Press, Philadelphia, PA, 1994.
3. J. Konhauser, D. Velleman, and S. Wagon, *Which Way Did the Bicycle Go?*, Dolciani Mathematical Expositions #18, MAA, Washington, D.C., 1996.