Kepler's Laws, Newton's Equations, Euler's Method

We describe the original motivating example for the development of calculus, Newton's proof that his simple Law of Universal Gravitation (inverse-square attraction) implies the complicated properties of planetary orbits observed by Kepler. This was the first unifying quantitative scientific theory, which gave a huge impetus to the scientific revolution and modern faith in human progress.

Kepler's Laws. These are *empirical* laws, summarizing telescope observations of Mars and other planets moving across the night sky over many years, imagined from a point of view "above" the plane of their orbits.

1. Each planet orbits in an ellipse with the sun at one focus.
2. A planet moves so its radius from the sun sweeps equal areas over equal times.
3. The period of a planet's orbit is proportional to the 3/2 power of its major axis.

These phenonmena are much more complicated and unexpected than the "perfect" uniform circular motion in the astronomical theories of the ancients, and we feel the need to explain them from simpler principles.

Newton's Equations. Newton explained motion in terms of forces pulling or pushing objects. The first example in physics is a spring stretched along the y-axis, pulling a mass at height y back to the origin with a force proportional to the displacement, but in the opposite direction: $F(y) = -y$. We can visualize this as a *vector field*, representing the force at the point y by a vector (arrow) of length y pointing to zero.

Newton's Second Law of Motion $F = ma$, which becomes $a = F$ for $m = 1$, says that a unit mass is pulled with acceleration equal to the force at its current position. Letting $y(t)$ be the height of the unit mass at time t, we have $a(t) = y''(t)$ and $F(y(t)) = -y(t)$, so Newton's Law says:

$$y''(t) = -y(t),$$

a second-order differential equation for unknown $y(t)$. The solutions are sinusoidal:

$$y(t) = a\cos(t) + b\sin(t),$$

describing the periodic bobbing of the mass around zero. Specifying the initial position $y(0)$ and velocity $y'(0)$ allows us to solve for the coefficients a, b, giving a unique solution to the equation, a well-defined prediction for the position at each time. (In real life, frictional forces will lead to deviations from this model.)

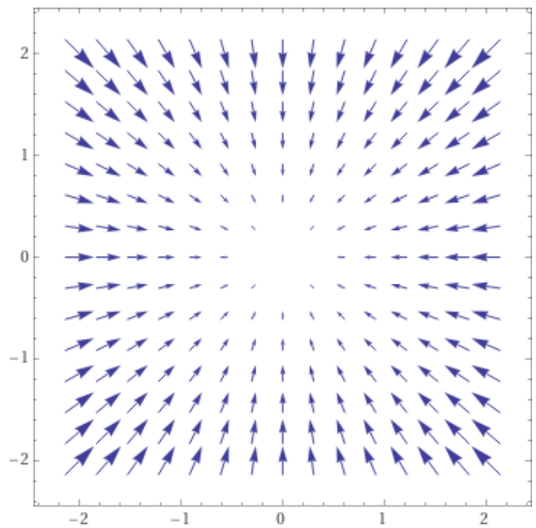
Our main concern will be motion in the (x, y)-plane, where we visualize a force field F (like gravity) as a vector at each point (x, y), pulling on any object at that point. We specify F in terms of the horizontal and vertical components (p, q) of the arrow from the point (x, y):

$$F(x, y) = (p(x, y), q(x, y)).$$

Notes by Peter Magyar `magyar@math.msu.edu`

Newton's Law of Gravitation asserts that a central massive object like the sun produces a force field pointing radially inward, with strength proportional to the inverse-square of the radial distance. A basic force field pointing inward is given by:

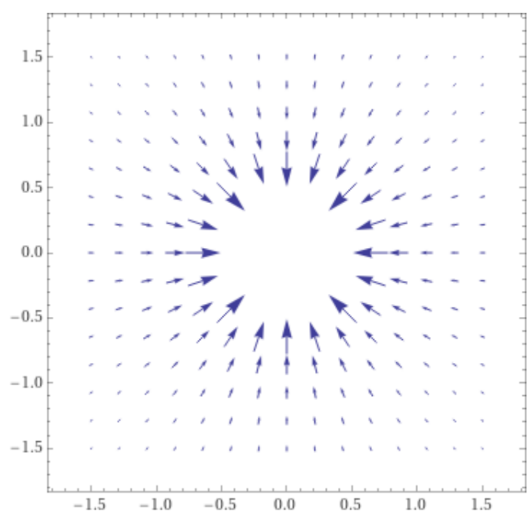
$$F(x,y) = -(x,y) = (-x,-y)$$



This has magnitude equal to the radius, getting stronger further from the center. Dividing by the radius $r = \sqrt{x^2+y^2}$ gives unit length; dividing by r^3 gives length $\frac{1}{r^2}$:

$$F(x,y) \;=\; -\frac{1}{r^3}(x,y) \;=\; \left(-\frac{x}{r^3},-\frac{y}{r^3}\right) \;=\; \left(\frac{-x}{(x^2+y^2)^{3/2}},\frac{-y}{(x^2+y^2)^{3/2}}\right).$$

This is Newton's force field (neglecting factors such as the gravitational constant G).



In planar motion, an object's position at time t is given by a parametric curve $(x(t), y(t))$, its velocity vector is $(x'(t), y'(t))$, and its acceleration is $(x''(t), y''(t))$. The Second Law $a = F$ becomes:

$$(x''(t), y''(t)) = F(x(t), y(t)).$$

For $F = (p, q)$, this gives a system of second-order differential equations:

$$\begin{cases} x''(t) &=& p(x(t), y(t)) \\ y''(t) &=& q(x(t), y(t)). \end{cases}$$

Substituting the inverse-square force, we arrive at Newton's equations for a planet orbiting the sun:

$$\begin{cases} x''(t) &=& \dfrac{-x(t)}{(x(t)^2{+}y(t)^2)^{3/2}} \\ y''(t) &=& \dfrac{-y(t)}{(x(t)^2{+}y(t)^2)^{3/2}}. \end{cases}$$

Newton's astonishing intellectual power discovered the physical laws of motion and gravitation, invented calculus to formulate these laws mathematically, and then solved the above equations to show they imply Kepler's Laws!

Numerical solutions: Euler's Method. Before discussing Newton's intricate algebraic calculations, we give some fairly direct numerical methods which approximately solve a very general class of differential equations, including Newton's. This is how engineers and scientists typically model physical phenomena.

Consider an unknown function $y = y(t)$ with given initial value $y(0)$ and satisfying:

$$y' \;=\; F(t, y),$$

where $F(t, y)$ is a given function of two variables. We fix a time increment Δt and sample points $t_i = i\,\Delta t$, then starting with $y(t_0) = y(0)$, we successively compute approximate values $y(t_1), y(t_2), y(t_3), \ldots$ as:

$$y(t{+}\Delta t) \;=\; y(t) + F(t, y(t))\,\Delta t.$$

That is, starting from the current point $t_i = t$, the graph follows the tangent line with slope $y'(t) = F(t, y(t))$ to reach the next point $t_{i+1} = t{+}\Delta t$. As $\Delta t \to 0$, the approximation $y(t)$ converges to the exact solution function, which is unique.

EXAMPLE: If $F(t, y) = F(t)$, the solution of $y'(t) = F(t)$ is $y(t) = y(0) + \int_0^t F(x)\,dx$, and Euler's Method just gives the left Riemann sum approximating this integral.

EXAMPLE: We usually compute $e = 2.7183\ldots$ by the Taylor series $e^t = 1{+}t{+}\frac{1}{2!}t^2{+}\cdots$ at $t = 1$. Another method is to numerically solve the exponential-growth equation:

$$y' = y \;\text{ with } y(0) = 1,$$

so that $y = e^t$ and $y(1) = e$. Here $F(t, y) = y$, so:

$$y(t{+}\Delta t) \;=\; y(t) + y(t)\,\Delta t.$$

For $\Delta t = \frac{1}{n}$, this gives the familiar formula $e \approx (1+\frac{1}{n})^n$. For $\Delta t = 0.1$, we compute:

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y(t)$	1.00	1.10	1.21	1.33	1.46	1.61	1.77	1.95	2.14	2.36	**2.59**

Errors accumulate, and the result is not very accurate. We improve it using the Midpoint Method, where we also compute half-way values to use for the next slope. After the given $y(0)$, we compute successive pairs $y(t-\frac{\Delta t}{2}), y(t)$ by the recurrence:

$$\begin{aligned} y(t+\tfrac{\Delta t}{2}) &= y(t) + F(t, y(t))\,\tfrac{\Delta t}{2} \\ y(t+\Delta t) &= y(t) + F(t+\tfrac{\Delta t}{2}, y(t+\tfrac{\Delta t}{2}))\,\Delta t. \end{aligned}$$

This takes twice the computation, but gives 30 times the accuracy:

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y(t-\frac{\Delta t}{2})$		1.050	1.160	1.282	1.417	1.565	1.730	1.911	2.112	2.334	2.579
$y(t)$	1.000	1.105	1.221	1.349	1.490	1.647	1.820	2.011	2.222	2.456	**2.714**

EXAMPLE: Approximate $y(t) = \sin(t)$ by solving the spring equation:

$$y'' = -y \quad \text{with} \quad y(0) = 0,\; y'(0) = 1.$$

This is a second-order differential equation of the form:

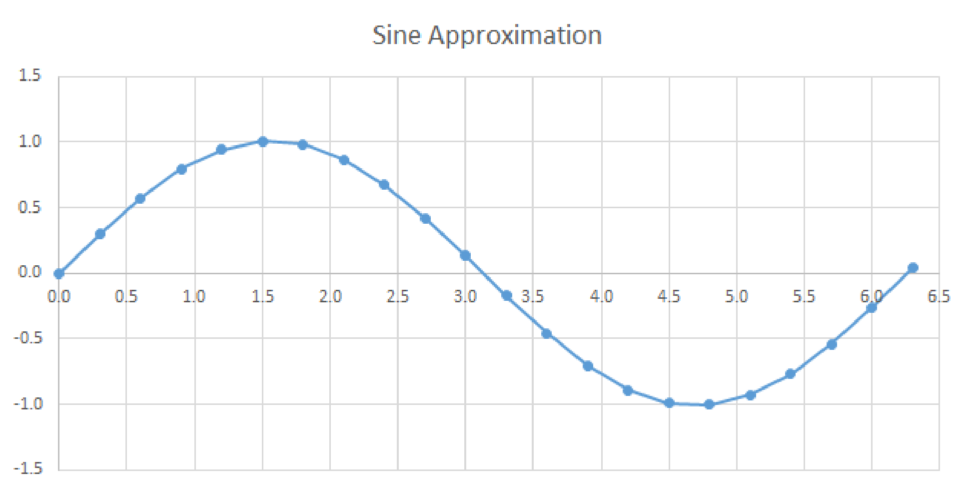
$$y'' = F(y) \quad \text{with given} \quad y(0), y'(0).$$

To solve, we compute lists of both $y(t)$ and $y'(t)$, updating y using its slope y', and updating y' using its slope $(y')' = F(y)$.

For accuracy, we use a version of the Midpoint Method called the Leapfrog Method. Instead of the initial value $y'(0)$, we use a staggered initial value $y'(\frac{\Delta t}{2}) = y'(0) + F(y(0))\frac{\Delta t}{2}$, then compute pairs $y(t), y'(t+\frac{\Delta t}{2})$, updating by:

$$\begin{aligned} y(t+\Delta t) &= y(t) + y'(t+\tfrac{\Delta t}{2})\,\Delta t \\ y'(t+\tfrac{3\Delta t}{2}) &= y'(t+\tfrac{\Delta t}{2}) + F(y(t+\Delta t))\,\Delta t. \end{aligned}$$

For $\Delta t = 0.3$, this gives the value of $\sin(t)$ correct to about 2 decimal places.



Numerical planetary model. To solve Newton's system of second-order differential equations for the trajectory $(x(t), y(t))$ in the force field $F(x,y) = (p(x,y), q(x,y))$,

$$\begin{cases} x''(t) &= p(x(t), y(t)) \\ y''(t) &= q(x(t), y(t)), \end{cases}$$

we again use the Leapfrog Method. Starting from intial values, we compute:

$$x(t),\ y(t),\ x'(t{+}\tfrac{\Delta t}{2}), y'(t{+}\tfrac{\Delta t}{2}),$$

updating by:

$$\begin{aligned} x(t{+}\Delta t) &= x(t) + x'(t{+}\tfrac{\Delta t}{2})\,\Delta t \\ y(t{+}\Delta t) &= y(t) + y'(t{+}\tfrac{\Delta t}{2})\,\Delta t \\ x'(t{+}\tfrac{3\Delta t}{2}) &= x'(t{+}\tfrac{\Delta t}{2}) + p(x(t{+}\Delta t), y(t{+}\Delta t))\,\Delta t \\ y'(t{+}\tfrac{3\Delta t}{2}) &= y'(t{+}\tfrac{\Delta t}{2}) + q(x(t{+}\Delta t), y(t{+}\Delta t))\,\Delta t. \end{aligned}$$

Here is the resulting orbit for $(x(0), y(0)) = (1, 0)$, $(x'(0), y'(0)) = (0, 0.6)$, $\Delta t = 0.045$.

Newton's Planetary Equations dt = 0.045

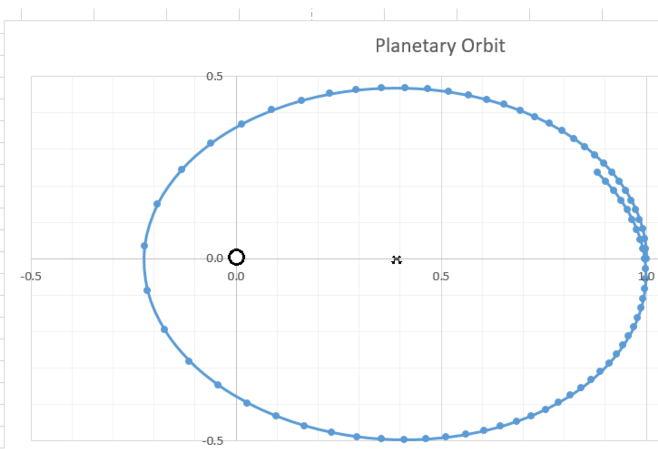
x(t+dt) = x(t) + x'(t+dt/2) dt p(x,y) = -x/(x^2+y^2)^(3/2)

y(t+dt) = y(t) + y'(t+dt/2) dt q(x,y) = -y/(x^2+y^2)^(3/2)

x'(t+3dt/2) = x'(t+dt/2) + p(x(t+dt),y(t+dt)) dt

y'(t+3dt/2) = y'(t+dt/2) + q(x(t+dt),y(t+dt)) dt

t	x(t)	y(t)	x'(t+dt/2)	y'(t+dt/2)
			0.000	0.600
0.00	1.000	0.000	-0.023	0.600
0.05	0.999	0.027	-0.068	0.599
0.09	0.996	0.054	-0.113	0.596
0.14	0.991	0.081	-0.158	0.593
0.18	0.984	0.107	-0.204	0.588
0.23	0.975	0.134	-0.250	0.581
0.27	0.963	0.160	-0.296	0.574
0.32	0.950	0.186	-0.344	0.564
0.36	0.935	0.211	-0.391	0.554
0.41	0.917	0.236	-0.440	0.541
0.45	0.897	0.261	-0.489	0.527
0.50	0.875	0.284	-0.540	0.510
0.54	0.851	0.307	-0.592	0.492

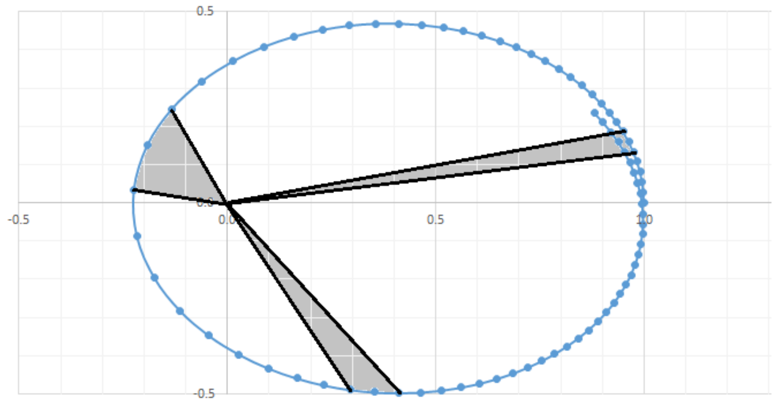


The orbit veers away from periodicity as the errors accumulate. Note how the Δt sample points crowd together at the slow far end of the orbit, but they are widely spaced at the close end where the sun's strong gravity speeds the planet quickly by.

A few spot-checks show our trajectories do approximately satisfy Kepler's Laws.

1. The orbit appears elliptical with semi-axes $a = 0.613$, $b = 0.483$ and center at $x = 0.387$. Thus the half-distance between foci is $c = \sqrt{a^2 - b^2} = 0.377$, and the foci are at $x = 0.387 \pm 0.377$, so the sun at $x = 0$ is a focus (error ± 0.01).

2. The three shaded sectors, each swept out over time $2\Delta t$, should have equal area.



The sectors are approximate triangles with two sides equal to the radius vectors $(x(t), y(t))$ and $(x(t{+}2\Delta t), y(t{+}2\Delta t))$. The triangle spanned by vectors (a, b) and (c, d) has area $A = \frac{1}{2}\det\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \frac{1}{2}(ad - bc)$, giving $A = 0.054$ for all three sectors. (Split the left sector into two triangles to account for its bulge.)

3. Our example has period and axis:

$$T_1 = 3.040, \quad a_1 = 0.613 \ .$$

Plugging initial conditions $(x(0), y(0)) = (0, 2)$, $(x'(0), y'(0)) = (0, 0.5)$ into the spreadsheet produces an orbit with:

$$T_2 = 12.275, \quad a_2 = 1.562,$$

giving close agreement between the period and axis ratios:

$$\frac{T_2}{T_1} = 4.038 \ \approx \ \frac{a_2^{3/2}}{a_1^{3/2}} = 4.067 \ .$$

Conserved quantities. To analyze the path of an object under Newton's equations

$$(x''(t), y''(t)) = F(x(t), y(t)) \quad \text{for} \quad F(x, y) = (p(x, y), q(x, y)) = -\tfrac{(x,y)}{(x^2+y^2)^{3/2}}$$

in terms of exact formulas, we first show the solutions are constrained by two quantities which remain constant over time: mechanical energy and angular momentum.

The idea of energy conservation is that the work to move an object against gravity gets "stored" in the object's new position, and this stored energy can be converted back into motion by falling. (This is the principle of hydroelectric power: a pump or the sun's evaporation can move water up into a reservoir, then this energy can be reclaimed as the water flows downhill and moves a turbine.) Computing the work done against our gravity field $F(r) = -\frac{1}{r^2}$ gives a *potential energy function*:

$$U(r) \ = \ -\int F(r)\,dr \ = \ \int \frac{1}{r^2}\,dr = -\frac{1}{r}, \qquad U(x, y) \ = \ -\frac{1}{\sqrt{x^2 + y^2}}.$$

The work needed to move a unit mass between two points is the change in U between them. Note U has a deep "gravity well" at the origin, so it would theoretically take infinite work to move from the center to a finite radius, though in fact our model only holds outside the radius of the sun. The flattening of U at distant points means that moving from a finite radius to infinity takes only finite work, which is why we can launch a spacecraft like Voyager out of the solar system.

As the field pulls an object, potential energy trades off with *kinetic energy*, energy of motion, defined as half the square of the speed: $\frac{1}{2}v(t)^2 = \frac{1}{2}(x'(t)^2+y'(t)^2)$. Their sum is the total *mechanical energy* which remains constant through time:

$$E(t) \stackrel{\text{def}}{=} \tfrac{1}{2}v(t)^2 + U(x(t),y(t)) \;=\; \tfrac{1}{2}(x'(t)^2+y'(t)^2) - \frac{1}{\sqrt{x(t)^2+y(t)^2}}.$$

Proof: For brevity, we write $E(t)$ as just E, etc. The Chain Rule gives:

$$E' \;=\; x'x'' + y'y'' + \frac{xx'+yy'}{(x^2+y^2)^{3/2}}.$$

Applying Newton's Equations $x'' = p(x,y)$, $y'' = q(x,y)$ gives:

$$E' \;=\; x'\frac{-x}{(x^2+y^2)^{3/2}} + y'\frac{-y}{(x^2+y^2)^{3/2}} + \frac{xx'+yy'}{(x^2+y^2)^{3/2}} \;=\; 0.$$

Since $E(t)$ has zero derivative, it is a constant, conserved quantity.

The second quantity conserved under Newton's equations is *angular momentum*, which measures the "swing" of the trajectory around the origin, defined as the radius times the component of velocity perpendicular to the radius. As the planet moves closer to the sun, its shorter radius is exactly compensated by a faster angular velocity.

Angular momentum can be computed as the area of the parallelogram spanned by the radius and velocity vectors. Linear algebra computes the area of a parallelogram spanned by two vectors from the origin to (a,b) and (c,d) by using a determinant:

$$\pm\text{Area} \;=\; \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;=\; ad - bc.$$

Here plus/minus is the sign of the angle $\theta \in [-\pi,\pi]$ turning from (a,b) to (c,d), indicating counterclockwise/clockwise rotation. Taking position vector $(a,b) = (x(t),y(t))$ and velocity vector $(c,d) = (x'(t),y'(t))$ gives angular momentum:

$$M(t) \stackrel{\text{def}}{=} \det\begin{bmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{bmatrix} \;=\; x(t)y'(t) - y(t)x'(t).$$

Again, a direct computation shows that $M'(t) = 0$, so $M(t)$ is constant over time:

$$M' \;=\; x'y' + xy'' - y'x' - yx'' \;=\; x\frac{-y}{(x^2+y^2)^{3/2}} - y\frac{-x}{(x^2+y^2)^{3/2}} \;=\; 0.$$

Reduction to polar coordinates. Orbits are much more conveniently described and analyzed in terms of the polar coordinate system, which specifies position in terms of radius $r = \sqrt{x^2+y^2}$ and angle $\theta = \arctan \frac{y}{x}$, so that:

$$(x, y) = (r\cos\theta, r\sin\theta).$$

The gravitational force can be written:

$$F(x,y) \;=\; -\frac{(x,y)}{(x^2+y^2)^{3/2}} \;=\; -\frac{1}{r^3}(r\cos\theta, r\sin\theta) \;=\; -\frac{1}{r^2}(\cos\theta, \sin\theta).$$

Here $(\sin\theta, \cos\theta)$ represents a field of unit vectors pointing radially outward, so F indeed points inward with magnitude $\frac{1}{r^2}$. We will also use the transverse vector field $(-\sin\theta, \cos\theta)$, unit vectors perpendicular to the radius in the direction of a counter-clockwise rotation.

Computing the radius and angle of each point of the parametric curve $(x(t), y(t))$ gives polar parametric functions:

$$r = r(t) = \sqrt{x(t)^2 + y(t)^2}, \quad \theta = \theta(t) = \arctan \tfrac{y(t)}{x(t)}.$$

In terms of these functions, the velocity vector is:

$$\begin{aligned}(x,y)' &= [\,r\,(\cos\theta, \sin\theta)\,]' \\ &= r'\,(\cos\theta, \sin\theta) + r\,((-\sin\theta)\theta', (\cos\theta)\theta') \\ &= r'\,(\cos\theta, \sin\theta) + r\,\theta'\,(-\sin\theta, \cos\theta).\end{aligned}$$

This decomposes velocity into its radial component $r' = r'(t)$, and perpendicular rotational component $r\theta' = r(t)\,\theta'(t)$. Thus we can re-calculate speed as:

$$v \;=\; \sqrt{(x')^2 + (y')^2} \;=\; \sqrt{(r')^2 + (r\theta')^2};$$

energy as:

$$E \;=\; \tfrac{1}{2}v^2 + U(r) \;=\; \tfrac{1}{2}(r')^2 + \tfrac{1}{2}r^2(\theta')^2 - \frac{1}{r};$$

and angular momentum as:

$$M \;=\; xy' - yx' \;=\; (r)(r\theta') \;=\; r^2\,\theta'.$$

To verify these formulas directly, we substitute $x = r\cos\theta$, $y = r\sin\theta$, and simplify.

Finally, Newton's equations become:

$$\begin{aligned}(x,y)'' &= [\,r'\,(\cos\theta, \sin\theta) + r\,\theta'\,(-\sin\theta, \cos\theta)\,]' \\ &= (r'' - r\,(\theta')^2)\,(\cos\theta, \sin\theta) + (2r'\theta' + r\theta'')\,(-\sin\theta, \cos\theta) \\ &\overset{\text{Newt}}{=} F(x,y) \;=\; -\tfrac{1}{r^2}(\cos\theta, \sin\theta).\end{aligned}$$

Equating the radial components on the two sides of Newton's Equation gives:

$$r'' - r\,(\theta')^2 \;=\; -\frac{1}{r^2}.$$

Since angular momentum $M = r^2\theta'$ is constant throughout a Newtonian trajectory, we can use it to eliminate $\theta' = M/r^2$. Newton's Equation becomes:

$$r'' \;=\; \frac{M^2}{r^3} - \frac{1}{r^2}.$$

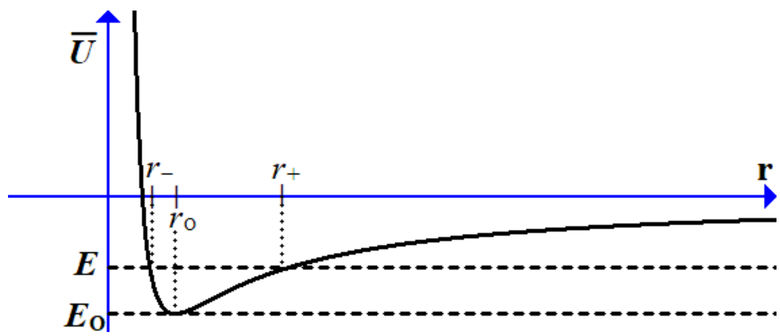
The same substitution also gives mechanical energy in terms of only r and r':

$$E \;=\; \tfrac{1}{2}(r')^2 + \frac{M^2}{2r^2} - \frac{1}{r},$$

Ignoring the rotation $\theta(t)$, we can visualize $r(t)$ as a one-dimensional Newtonian trajectory along the positive "radius axis" with effective force and potential energy:

$$\bar{F}(r) \;=\; \frac{M^2}{r^3} - \frac{1}{r^2}, \qquad \bar{U}(r) \;=\; \frac{M^2}{2r^2} - \frac{1}{r}.$$

We have put the rotational kinetic energy into the potential $\bar{U}(r)$ via $\frac{1}{2}r^2(\theta')^2 = \frac{M^2}{2r^2}$.



If $M \neq 0$, this has a positive spike instead of a gravity well near $r = 0$, where $\theta' = \frac{M}{r^2}$ is large and the planet swings around so fast it feels "centrifugal force" away from the sun. Orbital rotation is counterclockwise for $M > 0$ and clockwise for $M < 0$, while $M = 0$ means the planet moves along a line, the only way it can hit the sun.

For a given $M \neq 0$, an initial condition with negative energy $E < 0$ as shown gives a bounded planetary orbit rocking in the $U(r)$ trough $r_- \leq r(t) \leq r_+$, between perihelion and aphelion radii with $\bar{U}(r_-) = \bar{U}(r_+) = E$. Setting $E = E_\circ = -\frac{1}{2M^2}$, the minimum value of $\bar{U}(r)$, forces $r' = 0$, $r = r_\circ = M^2$, $\theta = \frac{M}{r_\circ^2}t = \frac{1}{M^3}t$, the unique circular orbit for the given M.*

Positive energy $E \geq 0$ occurs with super-escape velocity $v \geq \sqrt{2(\frac{1}{r} - \frac{M^2}{2r^2})}$, giving an infinite trajectory with $r(t) \to \infty$ as $t \to \pm\infty$, the path of an interstellar asteroid like ′Oumuamua transiting through the solar system.

*Kepler's third law for circular orbit: period $T = 2\pi M^3$, semi-major axis $r_\circ = M^2$, so $T = 2\pi r_\circ^{3/2}$.

Proof of Kepler's Laws.

1. An ellipse with semi-axes $a \geq b$ has standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and foci at $(\pm c, 0)$ for $c = \sqrt{a^2 - b^2}$. Shift the left focus to the orgin and substitute polar:

$$\frac{(x{-}c)^2}{a^2} + \frac{y^2}{b^2} \;=\; \frac{(r\cos\theta - c)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} \;=\; 1.$$

Solving for r, this gives a simple polar graph, setting $e = c/a$ and $\ell = b^2/a$:

$$r = \frac{\ell}{1 - e\cos\theta}.$$

Finally, tilting the major axis to the line $\theta = \alpha$ gives the general case:

$$r = \frac{\ell}{1 - e\cos(\theta{-}\alpha)}.$$

Here $0 \leq e < 1$ is called the *eccentricity*. For $e \geq 1$, this formula gives an infinite curve, a parabola or hyperbola with the origin at one focus.

We must show that a Newtonian trajectory $(r, \theta) = (r(t), \theta(t))$ satisfies the above equation for some e, ℓ, α. We eliminate the time variable t by using our conserved quantities, writing everything in terms of r as in our one-dimensional model above:

$$E \;=\; \tfrac{1}{2}(r')^2 + \tfrac{M^2}{2r^2} - \tfrac{1}{r} \quad\Longrightarrow\quad r' \;=\; \sqrt{2(E - \tfrac{M^2}{2r^2} + \tfrac{1}{r})}$$

$$M \;=\; r^2\theta' \quad\Longrightarrow\quad \theta' \;=\; \tfrac{M}{r^2}$$

$$\frac{d\theta}{dr} \;=\; \frac{d\theta/dt}{dr/dt} \;=\; \frac{\theta'}{r'} \;=\; \frac{M}{r^2\sqrt{2(E - \frac{M^2}{2r^2} + \frac{1}{r})}}$$

$$\theta \;=\; \int \frac{M}{r^2\sqrt{2(E{-}\frac{M^2}{2r^2}{+}\frac{1}{r})}}\,dr \;=\; \int \frac{1}{\sqrt{C^2{-}(\frac{1}{M}{-}\frac{M}{r})^2}}\,\frac{M}{r^2}\,dr \qquad \left[\begin{array}{c} C = \sqrt{2E + \frac{1}{M^2}} \\ u = \frac{1}{M} - \frac{M}{r} \\ du = \frac{M}{r^2}\,dr \end{array}\right]$$

$$=\; \int \frac{1}{\sqrt{C^2 - u^2}}\,du \;=\; -\cos^{-1}(\tfrac{u}{C}) + \alpha \;=\; -\cos^{-1}\Big(\tfrac{1}{\sqrt{1{+}2EM^2}}(1 - \tfrac{M^2}{r})\Big) + \alpha$$

where α is an arbitrary constant. Finally, solving for r:

$$r \;=\; \frac{M^2}{1 - \sqrt{1{+}2EM^2}\,\cos(\theta{-}\alpha)},$$

which is the desired formula with $\ell = M^2$ and $e = \sqrt{1 + 2EM^2}$.† This gives eccentricity $e < 1$ for $E < 0$, an ellipse orbit, including $e = 0$ for $E = E_\circ = -\frac{1}{2M^2}$, the case of the circle orbit above. We also get $e \geq 1$ for $E \geq 0$, a parabola or hyperbola orbit.

†The trajectory's shape and speed are determined by any two shape parameters: semi-major axis a, semi-minor axis b, aphelion $r_+ = a{+}c$, perihelion $r_- = a{-}c$, half-distance between foci $c = \sqrt{a^2{-}b^2}$, eccentricity $e = c/a$, latus rectum $\ell = b^2/a$, angular momentum $|M| = b/\sqrt{a}$, energy $E = -1/2a$, period $T = 2\pi a^{3/2}$. Attitude and phase are then determined by any two phase parameters: axis tilt α, initial position $x(0), y(0), r(0), \theta(0)$, initial velocity $x'(0), y'(0), r'(0), \theta'(0)$. There is a final discrete choice including clockwise/counterclockwise.

2. The area enclosed by a polar curve $r = r(\theta)$ with $\theta \in [\alpha, \beta]$ is $A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta$. For a polar parametric curve $(r(t), \theta(t))$ over time $t \in [t_1, t_2]$, the area formula undergoes the change of variable $\theta = \theta(t)$, $d\theta = \theta'(t)\, dt$, i.e.

$$A \;=\; \int_{t_1}^{t_2} \tfrac{1}{2} r(t)^2\, \theta'(t)\, dt.$$

When $(r(t), \theta(t))$ obey Newton's Equations, angular momentum $M = r(t)^2\, \theta'(t)$ is constant, so area over $t \in [a, b]$ is indeed proportional to the elapsed time:

$$A \;=\; \tfrac{1}{2} M (b - a).$$

3. The ellipse $r = \frac{\ell}{1 - e\cos(\theta - \alpha)}$ in the first law above has:

$$\ell = b^2/a = M^2 \qquad \Longrightarrow \qquad b \;=\; \sqrt{a\ell} \;=\; M a^{1/2}.$$

Its shape is a unit circle stretched horizontally by a and vertically by b, so its area is $A = \pi ab = \pi M a^{3/2}$. On the other hand, by the second law above, the area enclosed over a complete orbital period $t \in [0, T]$ is $A = \frac{1}{2} MT$. Thus:

$$A \;=\; \pi M a^{3/2} = \tfrac{1}{2} MT \qquad \Longrightarrow \qquad T \;=\; 2\pi a^{3/2}.$$

That is, the period is proportional to the 3/2 power of the semi-major axis.

Exact solution of Newton's Equations. The radial energy equation implies:

$$\tfrac{dr}{dt} \;=\; r' \;=\; \pm\sqrt{2(E - \bar{U}(r))} \;=\; \pm\sqrt{2(E - \tfrac{M^2}{2r^2} + \tfrac{1}{r})}\,,$$

which we can solve by separation of variables (assuming $r' \geq 0$):

$$t \;=\; \int dt \;=\; \int \frac{dr}{\sqrt{2(E - \frac{M^2}{2r^2} + \frac{1}{r})}} \;=\; \int \frac{r\, dr}{\sqrt{2Er^2 + 2r - M^2}} \;=\; \int \frac{a^{1/2}\, r\, dr}{\sqrt{-r^2 + 2ar - b^2}}\,,$$

where $E = -\frac{1}{2a} < 0$, $M^2 = \frac{b^2}{a}$. Denoting $c^2 = a^2 - b^2$ and substituting $u = r - a$:

$$\begin{aligned}
t \;&=\; \int \frac{a^{1/2} r}{\sqrt{c^2 - (r-a)^2}}\, dr \;=\; \int \frac{a^{1/2} u + a^{3/2}}{\sqrt{c^2 - u^2}}\, dr \\
&=\; -a^{1/2}\sqrt{c^2 - u^2} \;+\; a^{3/2} \sin^{-1} \tfrac{u}{c} \\
&=\; -\sqrt{-a(r{-}a{+}c)(r{-}a{-}c)} \;+\; a^{3/2} \sin^{-1} \tfrac{r-a}{c} \\
&=\; -\tfrac{1}{\sqrt{2|E|}} \sqrt{2Er^2 + 2r - M^2} \;-\; \tfrac{1}{(2|E|)^{3/2}} \sin^{-1} \tfrac{1+2Er}{\sqrt{1+2EM^2}}\,,
\end{aligned}$$

where r varies between perihelion $r_- = a - c$ and aphelion $r_+ = a + c$. Unfortunately, it is not possible to solve for r in terms of t, and hence to get $\theta = \int \theta'\, dt = \int \frac{M}{r^2}\, dt$ and $(x(t), y(t))$. There are no such formulas in terms of standard functions.

However, we do get time elapsed from $r(t_1) = r_1$ to $r(t_2) = r_2$ (assuming $r' > 0$):

$$t_2 - t_1 \;=\; \Big[\sqrt{-a(r{-}a{-}c)(r{-}a{+}c)} + a^{3/2} \sin^{-1}\big(\tfrac{r-a}{c}\big)\Big]_{r=r_1}^{r=r_2}.$$

In particular, we can re-prove Kepler's third law by computing the time elapsed from $r_1 = r_-$ to $r_2 = r_+$, getting the half-period $\frac{1}{2} T = \pi a^{3/2}$.