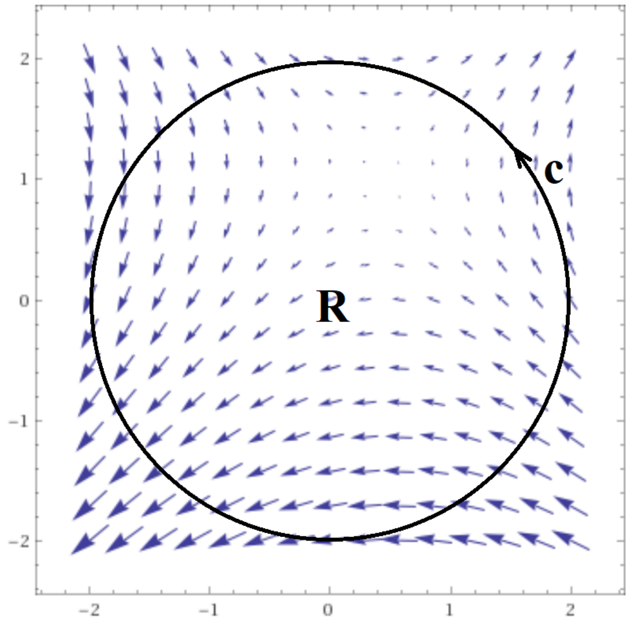
Name:__________________________

Math 254H **Midterm Exam** **Feb 26, 2020**

Don't write just answers: show your work, and justify steps by quoting theorems we have covered.

1. (30pts) The picture shows the gradient vector field $\mathbf{F}(x,y) = (y-1, x-\frac{1}{2}) = \nabla f(x,y)$ for some function $f(x,y)$; and also the region R enclosed by $\mathbf{c}$, the circle $x^2+y^2=4$.



a. On the picture, sketch a contour map of $f(x,y)$, including level curves through critical points.

b. On the picture, mark the absolute max/min points of $f(x,y)$ over the region R: max $\square$, min $\triangle$.

c. Determine the circulation integral of $\mathbf{F}$ around $\mathbf{c}$. Justify.

$$\oint \mathbf{F}(\mathbf{c}) \cdot d\mathbf{c} \ =$$

d. Determine the flux integral of $\mathbf{F}$ across $\mathbf{c}$, out of region R. Justify.

$$\oint \mathbf{F}(\mathbf{c}) \cdot d\mathbf{n} \ =$$

e. Use the Gradient Theorem to find a formula for $f(x,y)$.

2. (20pts) For each statement below, if it is true, give a proof using theorems we have covered; if false, give a counterexample.

a. *Claim:* For any differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, the curl of the gradient is zero: for all (x, y),

$$\text{curl}(\nabla f)(x, y) = 0.$$

b. *Claim:* For any differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, divergence of the gradient is zero: for all (x, y),

$$\text{div}(\nabla f)(x, y) = 0.$$

3. (25pts) Consider the parallelogram R with corners $(0,0),(1,0),(1,1),(2,1)$, enclosed by lines:

$$y=0,\quad y=1,\quad y=x,\quad y=x-1.$$

a. Parametrize the region R by a linear function $L(u,v)=(x(u,v),y(u,v))$ for $(u,v)\in[0,1]\times[0,1]$.

b. For a general function $f(x,y)$, set up its integral over R using the substitution $(x,y)=L(u,v)$.

$$\iint_R f(x,y)\,dx\,dy \;=$$

c. For a general function $f(x,y)$, set up its integral over R in the form $\int_{x=a}^{b}\int_{y=c(x)}^{d(x)} f(x,y)\,dy\,dx$.

$$\iint_R f(x,y)\,dy\,dx \;=$$

4. (25pts) A cycloid curve is traced by a point on the edge of a unit circle rolling on top of the x-axis.

a. Parametrize a cycloid, $\mathbf{c}(t) = (x(t), y(t))$. *Hint:* Radian angle turned = linear distance rolled.

b. Parametrize the region A under one arch of your cycloid: $G(t,s) = (x(t,s), y(t,s))$ for $(t,s) \in A^*$.

c. Determine the area under one arch of the cycloid:

$$\text{Area} \;=\; \iint_A 1\,dx\,dy \;=$$