Math 254H 4/6/2020 Electro magnetism

Coulomb's Law: point charge Q at origin produces electric field $\vec{E} = +\frac{Q\vec{v}}{4\pi|\vec{v}|^3}$

$\Downarrow$

Gauss Law: $\operatorname{div} E(\vec{v}) = \begin{cases} 0 & \vec{v} \neq \vec{0} \\ \infty & \vec{v} = 0 \end{cases}$

$\Downarrow$

$\iint_S \vec{E} \cdot d\vec{S} = \begin{cases} 0 & \text{if } \vec{0} \text{ not enclosed by } S \\ \cancel{4\pi} Q & \text{if } 0 \text{ is enclosed} \end{cases}$

Maxwell Law 1

charge distribution region $\tilde{R}$: $\delta(x,y,z)$

$\vec{E}(\vec{v}) = \iiint_{\tilde{R}} \frac{\delta(x,y,z)(\vec{v}-\vec{t})\,dR}{|\vec{v}-\vec{t}|^3}$

$\Downarrow$

$\boxed{\operatorname{div} \vec{E} = \cancel{+4\pi}\,\delta}$

$\Updownarrow$ Div thm (FTC)

$\iint_S \vec{E} \cdot d\vec{S}$

$\parallel$

$\cancel{4\pi}\iiint_R \overset{\operatorname{div} E}{\delta}\, dR$

$\operatorname{div} \vec{E} = \cancel{4\pi}\,\delta$

Rate of electric flux at a point
? = charge density at the point

Differential form

Total electric flux out of a surface
= total charge enclosed by surface

Gauss' Theorem (Div theorem)

$\iint_S E \cdot d\vec{S} = \cancel{4\pi}\iiint_R \delta\, dR$

Integral form

$\vec{E}$ electric field $\quad \delta$ = charge density

Maxwell 1

"Gauss Law, Coulomb's Law"

$$\operatorname{div} \vec{E} = \delta \iff \iint_S \vec{E} \cdot d\vec{S} = \iiint_R \delta \, dR$$

Maxwell 2

$\vec{B}$ magnetic field (electric current / moving charge)

"No magnetic monopoles"

$$\operatorname{div} \vec{B} = 0$$

Lorentz Force Law

Charged particle with charge q velocity $\vec{v}$ experience force $\vec{F}$ producing trajectory $\vec{c}(t)$

$$\vec{F} = q(\vec{E} + \underline{\vec{v} \times \vec{B}})$$

analogous to Newton's Law of Motion

$F = ma$

$\underline{G = a}$

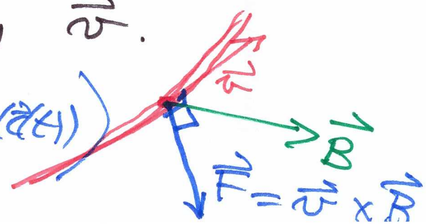
$$\vec{F}(\vec{c}(t)) = mc''(t)$$

$\vec{E}$ pushes/pulls particle like gravity.

$\vec{B}$ deflects the path of particle, perpendicular to its current velocity $\vec{v}$.

$$\vec{c}''(t) = \frac{\vec{F}}{m} = \frac{q}{m}\left(E(\vec{c}(t)) + c'(t) \times B(\vec{c}(t))\right)$$

Gravity: $\vec{c}''(t) = \vec{G}(\vec{c}(t))$



Maxwell 3

Faraday's Law

$\text{curl}\,\vec{E} = 0 \implies$ electric potential ϕ V

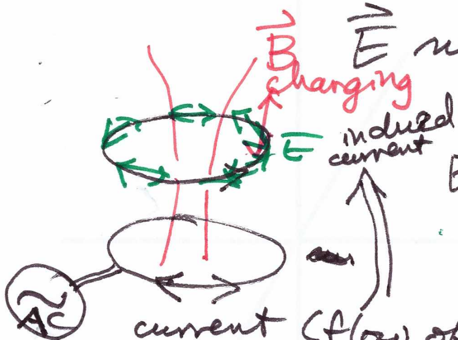
provided static situation (no motion of charges)

$$\text{curl}\,\vec{E} = -\frac{d\vec{B}}{dt}$$

in dynamic situation: charges move, $\vec{E} = \vec{E}(t)$

$\vec{B} = \vec{B}(t)$

$\vec{E}$ not conservative



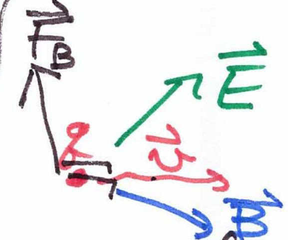
Electromagnetic induction

current (flow of charge) is sloshing back & forth

Maxwell 4

Math 254H 4/8/2020 Maxwell's Equations

Electric field $\vec{E}$
Magnetic field $\vec{B}$
induced by
given
charge density $\delta(x,y,z)$ (coulombs/cm^3)
current density $\vec{J}(x,y,z)$ (~~vector~~ field showing ~~volume~~ strength & direction of electric current)



Lorentz Force Law:
particle with charge q, velocity $\vec{v}$
experiences force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

direct force ($\vec{E}$) deflecting force ($\vec{v} \times \vec{B}$)

Simplified Maxwell's Eq.: Electrostatic case

(steady fields $\vec{E}, \vec{B}$ indep. of time)

derivative form

integral form

1. $\operatorname{div} \vec{E} = \delta$ $\longrightarrow$ $\displaystyle\iint_{\text{closed } S} \vec{E} \bullet d\vec{S} = \iiint_R \operatorname{div} \vec{E}\, dR \overset{(1)}{=} \iiint_R \delta\, dR$

   electric flux = charge enclosed

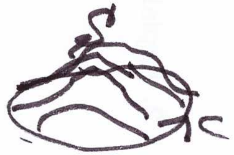
2. $\operatorname{div} \vec{B} = 0$ $\longrightarrow$ $\displaystyle\iint_{\text{closed } S} \vec{B} \bullet d\vec{S} = \iiint \operatorname{div} \vec{B}\, dR \overset{(2)}{=} 0$

   magnetic flux is zero
   magnetic field incompressible

3. $\operatorname{curl} \vec{E} = \vec{0}$ (always potential $\nabla\phi = \vec{E}$) $\longrightarrow$ $\displaystyle\oint_C \vec{E} \bullet d\vec{c} = \iint_S \operatorname{curl} \vec{E} \bullet d\vec{S} \overset{(3)}{=} 0$

   C closed
   circulation of electric field
   normal

4. $\operatorname{curl} \vec{B} = \vec{J}$



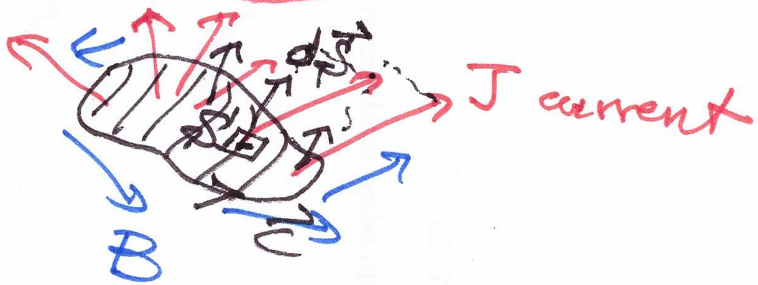
④ curl $\vec{B} = \vec{J}$ ⟶ $\int_{\vec{c}} \vec{B}\cdot d\vec{c} = \iint_{S} \text{curl}\,\vec{B}\cdot d\vec{S}$

circulation of B
around closed curve $\vec{c}$

$\overset{④}{=} \iint_{S} \vec{J} \cdot d\vec{S}_{\text{normal}}$

= flux of J enclosed by $\vec{c}$

= total current enclosed by $\vec{c}$



Simple solutions:

⊙ Point-charge: $\vec{E}(\cancel{\vec{x}}) = \dfrac{Q\vec{x}}{|\vec{x}|^3}$

$\vec{x} = (x, y, z)$ $\qquad \vec{B}(\vec{x}) = \vec{0}$

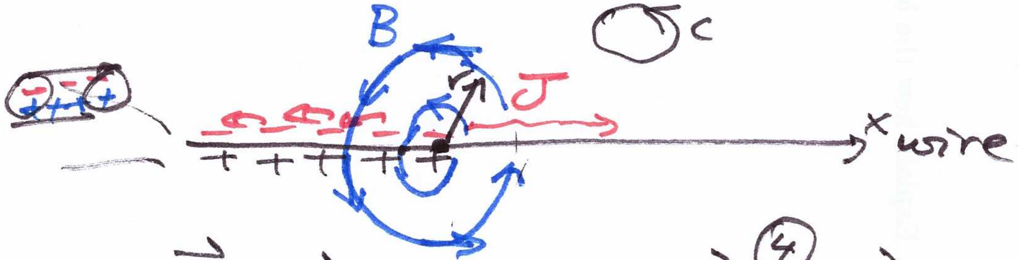
No current

$\vec{J}$ concentrated along wire (x-axis) → infinite density.

$\vec{J}(\vec{x}) = 0$ outside x-axis

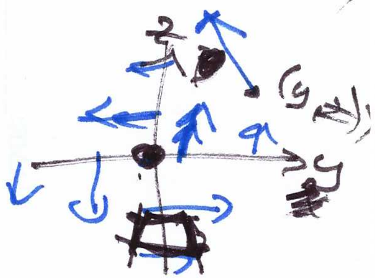
$\vec{B}(x,y,z) = \left(0, \dfrac{-z}{y^2+z^2}, \dfrac{y}{y^2+z^2}\right)$

⊙ Current along wire



$\vec{E} = \vec{0}$ $\qquad$ curl $\vec{B} \overset{④}{=} \vec{J}$

$\vec{B}(0, y, z) = \dfrac{(0, -z, y)}{(y^2+z^2)}$



Math 254H 4/10/2020 Finish Maxwell Equations.

Maxwell Equations for Electrostatic case: $\vec{E}$ & $\vec{B}$ are static indep of time

Charge density $\delta(x,y,z)$, current density $\vec{J}(x,y,z)$ generate electric field $\vec{E}$, magnetic field $\vec{B}$ producing force field $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ on particle with charge q & velocity $\vec{v}$ (Lorentz Force Law)

1. $\operatorname{div} \vec{E} = \delta$ (flux of $\vec{E}$ equals charge)
2. $\operatorname{div} \vec{B} = 0$ ($\vec{B}$ has zero flux)
3. $\operatorname{curl} \vec{E} = \vec{0}$ ($\vec{E}$ is conservative)
4. $\operatorname{curl} \vec{B} = \vec{J}$ ($\vec{B}$ rotate around currents)

Simplest solutions:

(a) Point charge (Coulomb field)

$$\vec{E}(\vec{x}) = \frac{Q\vec{x}}{|\vec{x}|^3}$$ $\delta = Q$ charge at $\vec{x} = \vec{0}$, inverse-square, $\vec{x} = (x,y,z)$

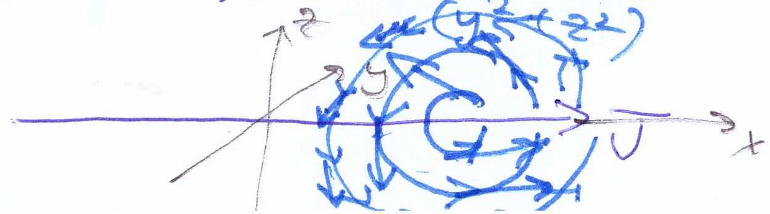
$$\vec{B}(\vec{x}) = \vec{0}$$ $\vec{J} = 0$

(b) Linear current

$\delta = 0$, $\vec{J}$

$\vec{E} = \vec{0}$, $J =$ ~~line~~ current in wire

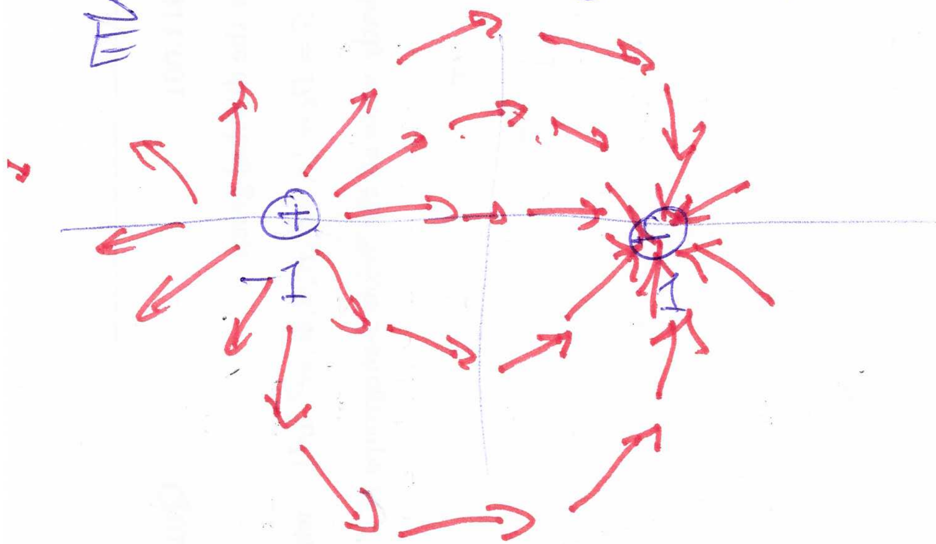
$$\vec{B}(x,y,z) = \frac{(0,-z,y)}{(y^2+z^2)}$$



More solutions:

- ~~Pos~~ Pos & Neg charge $\vec{B} = \vec{0}$

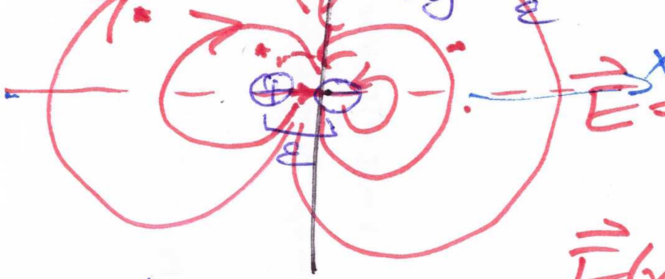
$\vec{E}$



- Electric dipole: $\vec{B} = 0$

push charges together (dist ε)

scale by $\frac{1}{\varepsilon}$



$$\vec{E} = \frac{(3xy, -x^2+2y^2)}{(x^2+y^2)^{5/2}}$$

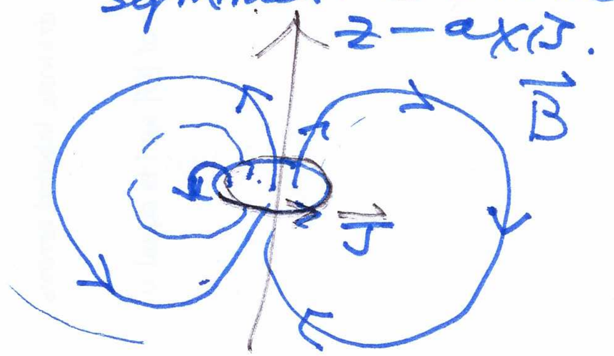
$\vec{E}(x,y,0)$

(Exercise)

- Magnetic dipole

$B(x,y,0) = \vec{B} = ($ same formula $)$

symmetric around z-axis.



Full Maxwell Equations $\quad \delta$ & J vary with time

$\vec{E} \quad \vec{B}$

$\vec{E}(x, y, z; t)$

$\vec{B}(x, y, z; t)$

Gauss

1. $\text{div}\, \vec{E} = \delta$

No magnetic monopole

2. $\text{div}\, \vec{B} = \vec{0}$

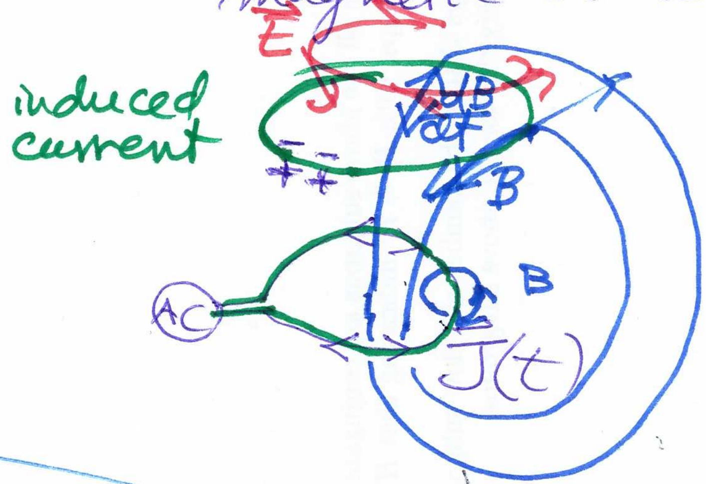
3. Faraday

$$\text{curl}\, \vec{E} = \frac{d\vec{B}}{dt}$$

4. Ampere

$$\text{curl}\, \vec{B} = \underbrace{\vec{J}}_{\text{current}} + \underbrace{\frac{d\vec{E}}{dt}}_{\text{"displacement current"}}$$

Faraday's Experiment: magnetic induction $\vec{E} = 0$



changing electric field equivalent to moving charge

Reference: George Arfken
Mathematical Methods for Physicists.