Math 254H 4/17/2020 General differtial forms.

φ k-form on $\mathbb{R}^n$: field of k-linear functions over $\mathbb{R}^n$

$\vec{a} \in \mathbb{R}^n$, $\varphi_{\vec{a}} : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \longrightarrow \mathbb{R}$

$\varphi_{\vec{a}}(\vec{h}_1, \ldots, \vec{h}_k)$, ~~linear~~ linear in each $\vec{h}_i$, alternating $\vec{h}_i \leftrightarrow \vec{h}_j$.

- Basis of coordinate forms $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$f = x_i$ is a function on $\mathbb{R}^n \Longrightarrow$ coordinate 1-form dx_i

$dx_i(\vec{h}) = dx_i(h_{x_1}, \ldots, h_{x_n}) = h_{x_i}$

$$\varphi = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \cdots i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

coeff function. $f_{i_1 \cdots i_k}(x_1, \ldots, x_n)$

- Integrate over k-dim surface $S \subset \mathbb{R}^n$

$P : S^* \longrightarrow S$, $S^* \subset \mathbb{R}^k$, $S \subset \mathbb{R}^n$

$P(u_1, \ldots, u_k) = (x_1(u_1 \ldots u_k), \ldots, x_n(u_1 \ldots u_k))$

$$\int_S \varphi = \int_{u_1} \cdots \int_{u_k} \varphi\left(\frac{\partial P}{\partial u_1}, \ldots, \frac{\partial P}{\partial u_k}\right) du_1 \cdots du_k$$

• Exterior derivative: φ is k-form $\longrightarrow$ $d\varphi$ $(k+1)$-forms

$$d\varphi_{\vec{a}}(\vec{h}_1,\dots,\vec{h}_k) = \left(\begin{array}{c}\text{rate of integral of } \varphi \\ \text{over } k\text{-dim paralellopipfied} \\ \text{spaned by } \vec{h}_1,\dots,\vec{h}_k\end{array}\right) \Big/ \begin{array}{c}(k+1)\text{ dim volume} \\ \text{enclosed}\end{array}$$

$$d\left(\sum_{i_1<\dots<i_k} f_{i_1\dots i_k}\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\right)$$

$$= \sum_{i_1<\dots<i_k} (df_{i_1\dots i_k})\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}.$$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \dots + \frac{\partial f}{\partial x_n}dx_n$$

• <u>Stokes Theorem</u>: (Fundamental Theorem of Calculus)

k-dim closed S = boundary of $(k+1)$-dim R $\quad (S = \partial R)$

k-form φ over $\mathbb{R}^n$

$$\overbrace{\int_{S=\partial R} \varphi}^{k\text{-dim int}} = \overbrace{\int_R d\varphi}^{(k+1)\text{-int}}$$

Special cases: Single-Var FTC, Grad Thm, Curl Thm, Div Thm

Divergence Thm in $\mathbb{R}^3$ in terms of diff. forms.

$$\iint_{S=\partial R} \vec{F} \cdot d\vec{S} \quad \text{LHS} \qquad = \qquad \iiint_R (\operatorname{div} \vec{F}) \, dR \quad \text{RHS}$$

total flux of $\vec{F}$ out of closed surface S

integral of rate of flux over region R enclosed by S

$$\vec{F} = (p, q, r)$$

$\updownarrow$ Hodge-star duality.

2-form $\eta = p \; dy \wedge dz - q \; dx \wedge dz + r \; dx \wedge dy$

$(+q \; dz \wedge dx)$

flux int

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\underline{S}} \eta$$

$$\iint_{S^*} \vec{F}(P(u,v)) \cdot \underbrace{\left(\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}\right) du \, dv}_{d\vec{S}} = \iint_{S^*} \eta_{P(u,v)}\left(\frac{\partial P}{\partial u}, \frac{\partial P}{\partial v}\right) \underline{du \, dv}$$

Don't use dot product

$P : S^* \longrightarrow S$

$P(u,v) = (x, y, z)$

RHS of Divergence thm

$$\iiint_R (\operatorname{div} \vec{F})\, dR$$

2-form

$$\vec{F} = (p, q, r) \longleftrightarrow \eta = p\, dy\, dz - q\, dx\, dz + r\, dx\, dy$$

$dy\, dx = -dx\, dy$

rate of flux

divergence

exterior derivative

rate of $\iint \eta$ out of box $(\vec{h}_1, \vec{h}_2, \vec{h}_3)$

3-form

$$\operatorname{div} \vec{F} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}$$

$$= dp \wedge dy \wedge dz - dq \wedge dx \wedge dz + dr \wedge dx \wedge dy$$

1-form

$$\left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz\right) \wedge dy \wedge dz + \ldots$$

$$= \frac{\partial p}{\partial x} dx \wedge dy \wedge dz + \cancel{\frac{\partial p}{\partial y} (dy) \wedge (dy \wedge dz)}_{\to 0} + \cancel{\frac{\partial p}{\partial z} dz \wedge dy \wedge dz}_{\to 0} + \ldots$$

$$- \frac{\partial q}{\partial y} dy \wedge dx \wedge dz$$

$$+ \frac{\partial r}{\partial z} dz \wedge dx \wedge dy$$

alternating

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = -\varphi_2 \wedge \varphi_1 \wedge \varphi_3$$

1-forms

$$dy \wedge dy \wedge dz = -dy \wedge dy \wedge dz = 0$$

$$= \frac{\partial p}{\partial x} dx \wedge dy \wedge dz + \frac{\partial q}{\partial y} dx \wedge dy \wedge dz + \frac{\partial r}{\partial z} dx \wedge dy \wedge dz = (\operatorname{div} \vec{F})\, dx \wedge dy \wedge dz$$

$$\iiint_R f(x,y,z) \overbrace{dx\,dy\,dz}^{\lim \Delta x \Delta y \Delta z} = \iiint_R \overbrace{f(x,y,z)\, dx \wedge dy \wedge dz}^{3\text{-form}}$$

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{flux}} \leftarrow \vec{F} \longrightarrow p\, dy\, dz - q\, dx\, dz + r\, dx\, dy$$

$$\int \vec{F} \cdot d\vec{c} \leftarrow \vec{F} \longrightarrow p\, dx + q\, dy + r\, dz$$