Math 254 H 4/3/2020

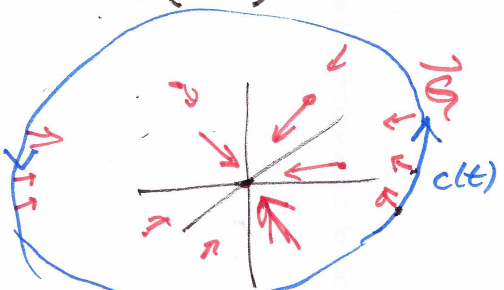
In $\mathbb{R}^3$, gravitational field $\vec{G}(x,y,z) = (p, q, r)$

Law of Gravitation: point-mass M at origin $\vec{O}$

$$\vec{G}(\vec{v}) = -M \frac{\vec{v}}{|\vec{v}|^3} \longrightarrow$$ inverse-square law

$$|\vec{G}| = \frac{M}{|\vec{v}|^2}$$

direction is toward point-mass at origin



Newton's Force Law (2nd Law) force on mass m, moves along trajectory $\vec{c}(t) = (x(t), y(t), z(t))$

$$\vec{F} = m\vec{G} = -mM \frac{\vec{v}}{|\vec{v}|^3}$$

$$\vec{F}(\vec{c}(t)) = m\,\vec{c}\,''(t)$$ $F = ma$

i.e. $$\vec{c}\,''(t) = \vec{G}(\vec{c}(t)) \Longrightarrow$$ trajectories: ellipse or ~~parabola~~ or hyperbola

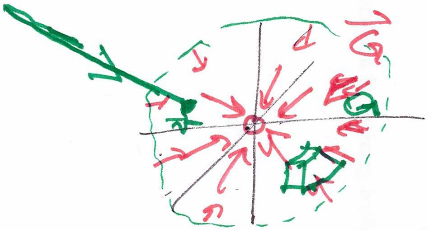
More generally:

sums planets $i = 1, 2, 3, \ldots$ $$\vec{G}(\vec{v}) = -\sum_i M_i \frac{\vec{v} - \vec{c}_i(t)}{|\vec{v} - \vec{c}_i(t)|^3}$$ $\vec{c}_i$ = pos of i^{th} mass

How does $\vec{G} = -\frac{\vec{v}}{|\vec{v}|^3}$ relate to derivs & integrals?

Potential fun $\vec{G} = \nabla g$? (physics potential $\phi = -g$)

$\nabla \vec{G} = -\phi$



Conservative Vect Field Thm

Exists pot g $\iff$ path-indep

$\iff$ circulation-free

$\iff$ irrotational curl $\vec{G} = 0$

Compute: <u>any radial</u>: $G(\vec{v}) = f(|\vec{v}|)\,\vec{v} \Rightarrow$ curl $\vec{G} = 0$ ✓

fun of radius

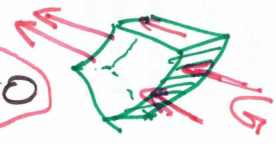
$g(\vec{v}) = -\frac{1}{|\vec{v}|}$ ← find by:

$g(\vec{v}) - g(\vec{a}) = \int_{\vec{c}} \vec{G} \cdot d\vec{c}$ $\quad \vec{c}(0) = \vec{a}$, $\vec{c}(1) = \vec{v}$

$\phi(\vec{v}) = +g(\vec{v}) = +\frac{1}{|\vec{v}|}$ "gravity well"

Other properties of $\vec{G}$? Flux: $\operatorname{div} G = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} = 0$

Only <u>inv-square</u> $\vec{G}$ has div = 0



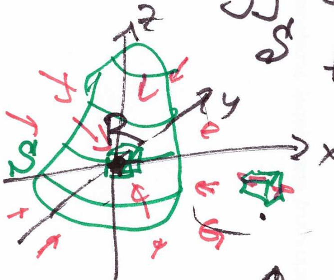
Thus $\vec{G} = \frac{\vec{v}}{|\vec{v}|^3}$ has $\operatorname{div}(G) = \begin{cases} 0 & \vec{v} \neq \vec{0} \\ \infty & \vec{v} = \vec{0} \end{cases}$

so $\iint_S \vec{G} \cdot d\vec{S} \neq \iiint_R \underbrace{\operatorname{div} G}_{0} \, dR = 0$

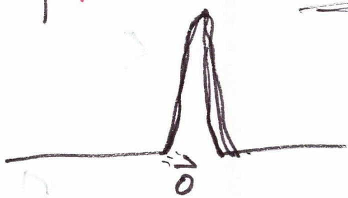
flux integral

R enclosed by S

$0 \leftarrow$ except ∞ at origin.



$\operatorname{div} \vec{G}$ = "delta function"

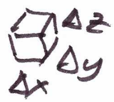


$= \begin{cases} 0 \text{ except at } \vec{v} = 0 \\ \text{so neg infinite at } \vec{v} = 0 \\ \text{that } \iiint_R \operatorname{div} \vec{G} \neq 0 \end{cases}$

In general, if mass density $\rho(x, y, z)$ g/m³ produces $\vec{G}(x, y, z)$:

$$\operatorname{div} \vec{G} = -\rho$$

Reformulation Newton's inverse-square Law of Gravitation



mass $= \rho(x, y, z) \, \Delta x \, \Delta y \, \Delta z$