

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ $f(x, y, z)$ scalar value

$\nabla f = \vec{F}$ gradient vector field

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ direction of max increase of f near (x, y, z)
 $\nabla f(x, y, z)$

Linear approx:

$f(\vec{v} + \vec{h}) \approx f(\vec{v}) + \nabla f(\vec{v}) \cdot \vec{h}$
base pt $\vec{v} = (x, y, z)$ increment \vec{h} base value $f(\vec{v})$ increment of f

Conservative Vector Field Theorem:

The following are equivalent for vector field \vec{F}

- ① \vec{F} conservative: exists potential, $\vec{F} = \nabla f$
- ② \vec{F} path independent: $\int \vec{F}(\vec{c}_1) \cdot d\vec{c}_1 = \int \vec{F}(\vec{c}_2) \cdot d\vec{c}_2$ provided $\vec{c}_1(0) = \vec{c}_2(0)$ and $\vec{c}_1(1) = \vec{c}_2(1)$
- ③ \vec{F} circulation-free: $\oint \vec{F}(\vec{c}) \cdot d\vec{c} = 0$ provided $\vec{c}(0) = \vec{c}(1)$ closed loop
- ④ \vec{F} irrotational: $\text{curl } \vec{F} = 0$?

What is curl ? $\vec{F} = (P, Q, R)$ $\text{curl } \vec{F} = \begin{pmatrix} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \\ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \end{pmatrix}$ 2-dim

call proof of G.V.F.T:

(2)

\Rightarrow (ii) Pot fun $\vec{F} = \nabla f$

then $\int \vec{F}(\vec{c}_1) \cdot d\vec{c}_1 = \int \vec{\nabla} f(\vec{c}) \cdot d\vec{c}$

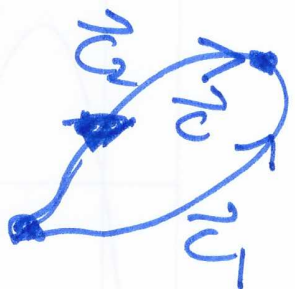
$$= f(\vec{c}_1(1)) - f(\vec{c}_1(0))$$

(Gradient Theorem)

$$= f(\vec{c}_2(1)) - f(\vec{c}_2(0))$$

$$= \int \vec{F}(\vec{c}_2) \cdot d\vec{c}_2$$

\Rightarrow (iii) If F is path-indep,



in \mathbb{R}^3

$$\oint \vec{F}(\vec{c}) \cdot d\vec{c} = \int \vec{F}(\vec{c}_1) \cdot d\vec{c}_1$$

$$- \int \vec{F}(\vec{c}_2) \cdot d\vec{c}_2 = 0$$

$$F = (p, q) \text{ in } \mathbb{R}^2$$

$$\text{curl } F = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \text{ Why?}$$

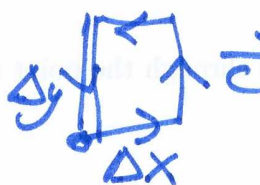
Want to define curl \vec{F} ③

(which is 0 if \vec{F} is irrotational-free)

Motivation, geometric idea?

2-dim $\vec{F} = (p, q)$

$$\text{curl } \vec{F}(x, y) = \lim_{\vec{c} \rightarrow (x, y)} \frac{\oint \vec{F}(\vec{c}) \cdot d\vec{c}}{(\text{area inside } \vec{c})}$$



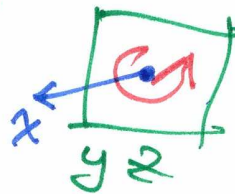
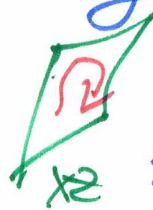
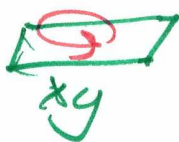
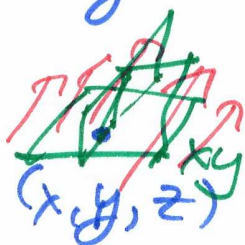
$$= \lim_{\Delta x \Delta y \rightarrow 0} \frac{\oint \vec{F}(\vec{c}) \cdot d\vec{c}}{\Delta x \Delta y}$$

Rate of circulation
per area enclosed,
near point (x, y)

For $\vec{F} = (p, q, r)$ 3-dim vect fld.

need to compute rate of circulation

in any orientation, any plane



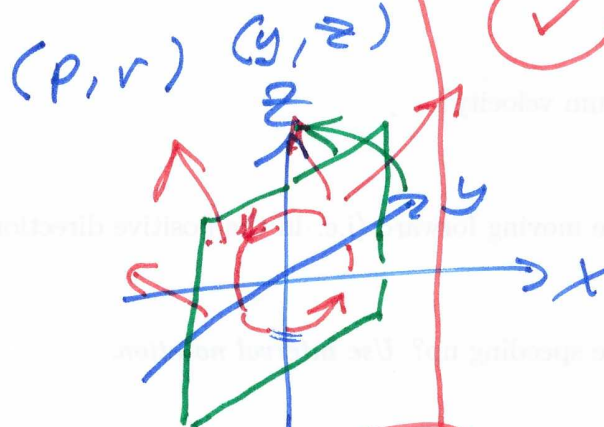
$$\vec{F} = (p, q, r)$$

(4)

Define $\text{curl } \vec{F}$ a vector:

$$\text{curl } \vec{F} = (\underbrace{\text{curl}_{yz} \vec{F}}_{\text{around } x\text{-axis}}, \underbrace{-\text{curl}_{xz} \vec{F}}_{y\text{-axis}}, \underbrace{\text{curl}_{xy} \vec{F}}_{z\text{-axis}})$$

$$\text{curl}_{yz} \vec{F} = \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}$$

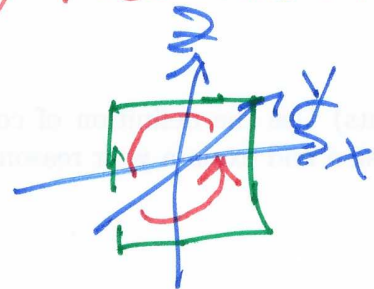


✓ correct orientation

✗ Reverse orientation

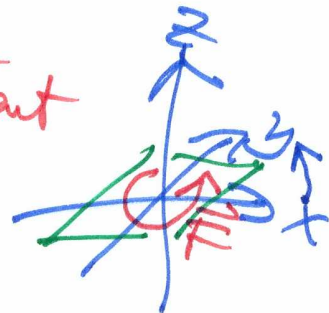
$$\text{curl}_{xz} \vec{F} = \frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}$$

$$(p, q, r)$$



$$\text{curl}_{xy} \vec{F} = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

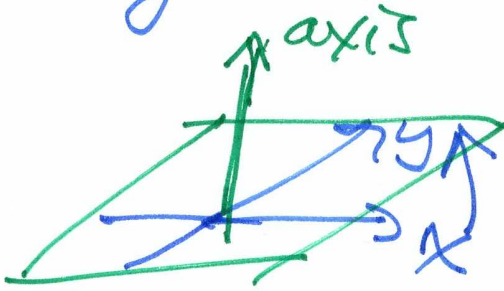
correct orient ✓



(p, q, ~~r~~) vars (x, y) z = const

Orientation (signs)

Right hand rule



Compute $\vec{F} = (p, q, r)$

curl $\vec{F} =$ rate of circulation
rot around

~~$\frac{\partial q}{\partial y} - \frac{\partial p}{\partial x}$~~ x y z - axes

$$\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}, -\frac{\partial r}{\partial x} + \frac{\partial p}{\partial z}, \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right)$$

Prop: Rate of circulation around
any ^{unit} vector \vec{q} (in plane $\perp \vec{q}$)
 $|\vec{q}| = 1$

$= (\text{curl } \vec{F}) \cdot \vec{q}$ "directional curl"

Ex: $\vec{q} = \vec{z} = (0, 0, 1)$ gives $\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}$

⑥

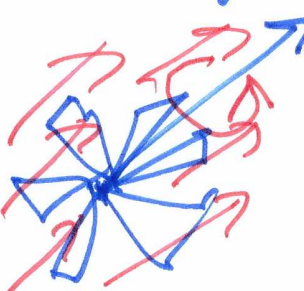
Analogy: Directional derivative
of $f(x, y, z)$. $f: \mathbb{R}^3 \rightarrow \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

3 coordinate rates of change

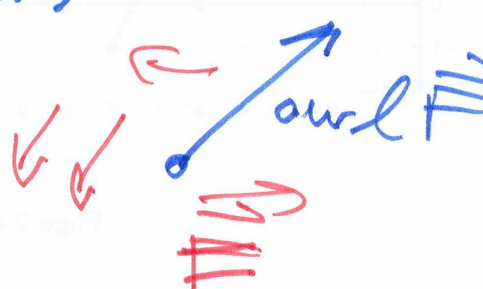
• \vec{q} unit vector

$$\frac{\partial f}{\partial \vec{q}} = \lim_{h \rightarrow 0} \frac{f(\vec{v} + h\vec{q}) - f(\vec{v})}{h}$$


$$\stackrel{!}{=} (\nabla f) \cdot \vec{q}$$

∇f = dir of max increase

curl \vec{F} = dir of maximum rotation



= primary rotation axis of \vec{F} at (x, y, z)



Name

Section

Instructor

INSTRUCTIONS

$$\begin{aligned} \vec{F} &= (p, q, r) \\ \text{curl } \vec{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (p, q, r) \\ &= \nabla \times \vec{F} \end{aligned}$$

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- Do not read or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the professor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room on the back of a page, you must indicate if you do so on the back of a page to be graded.
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