Math 254H Wed 3/18/2020

$f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z)$ scalar value

$\nabla f = \vec{F}$ gradient vector field

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$ direction of max increase of f near (x,y,z)

$\nabla f(x,y,z)$

Linear approx:

$$f(\vec{v} + \vec{h}) \approx \underbrace{f(\vec{v})}_{\text{base value}} + \cancel{\nabla f(x,y,z)} \underbrace{\nabla f(\vec{v}) \cdot \vec{h}}_{\text{increment of } f}$$

base pt $\vec{v} = (x,y,z)$, increment $\vec{h}$

Conservative Vector Field Theorem:

The following are equivalent for vector field $\vec{F}$

(i) $\vec{F}$ conservative: exists potential, $\vec{F} = \nabla f$

(ii) $\vec{F}$ path independent: $\int \vec{F}(\vec{c}_1) \cdot d\vec{c}_1 = \int \vec{F}(\vec{c}_2) \cdot d\vec{c}_2$ provided $\vec{c}_1(0) = \vec{c}_2(0)$ and $\vec{c}_1(1) = \vec{c}_2(1)$

(iii) $\vec{F}$ circulation-free: $\oint \vec{F}(\vec{c}) \cdot d\vec{c} = 0$ provided $\vec{c}(0) = \vec{c}(1)$ closed loop

(iv) $\vec{F}$ irrotational: curl $\vec{F} = 0$?

What is curl?

$\vec{F} = (P,Q,R)$ ... is ...

2-dim: $F = (P,Q)$ curl $F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

call proof of C.V.F.T :

$\Rightarrow$ (ii) Pot fun $\vec{F} = \nabla f$

en $\int \vec{F}(\vec{c}_1)\cdot d\vec{c}_1 = \int \vec{\nabla} f(\vec{c}_1)\cdot d\vec{c}_1$

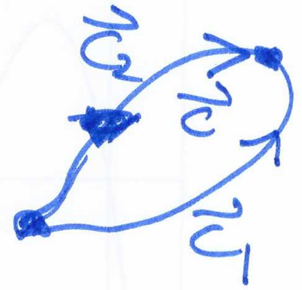
$= f(\vec{c}_1(1)) - f(\vec{c}_1(0))$

(Gradient theorem)

$= f(\vec{c}_2(1)) - f(\vec{c}_2(0))$

$= \int \vec{F}(\vec{c}_2)\cdot d\vec{c}_2$

$\Rightarrow$ (iii) If F is path-indep,

n  in $\mathbb{R}^3$

$\oint \vec{F}(\vec{c})\cdot d\vec{c} = \int \vec{F}(\vec{c}_1)\cdot d\vec{c}_1 - \int \vec{F}(\vec{c}_2)\cdot d\vec{c}_2 = 0$

$F = (p, q)$ in $\mathbb{R}^2$

curl $F = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$ Why?

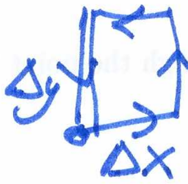
Want to define curl $\vec{F}$

(which is 0 if $\vec{F}$ is circulation-free)

Motivation, geometric idea?

2-dim $\quad \vec{F} = (p,q)$

$$\operatorname{curl} \vec{F}(x,y) = \lim_{\vec{c} \to (x,y)} \frac{\oint \vec{F}(\vec{c}) \cdot d\vec{c}}{(\text{area inside } \vec{c})}$$

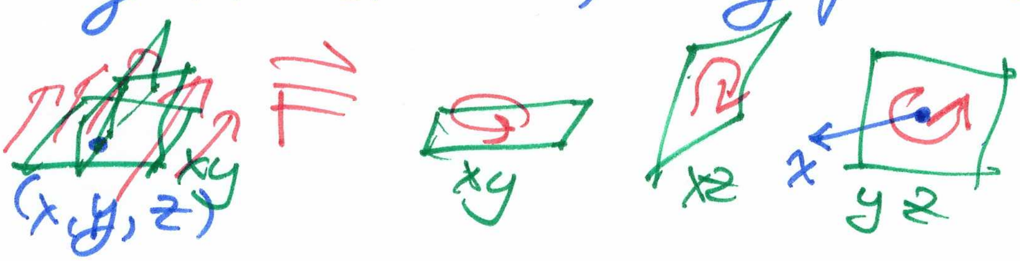


$$\vec{c} = \lim_{\Delta x \Delta y \to 0} \frac{\oint \vec{F}(\vec{c}) \cdot d\vec{c}}{\Delta x \Delta y}$$

Rate of circulation per area enclosed, near point (x,y)

For $\vec{F} = (p,q,r)$ 3-dim vect fld.

need to compute rate of circulation in any orientation, any plane



$\vec{F} = (p, q, r)$ (4)

Define curl $\vec{F}$ a vector:

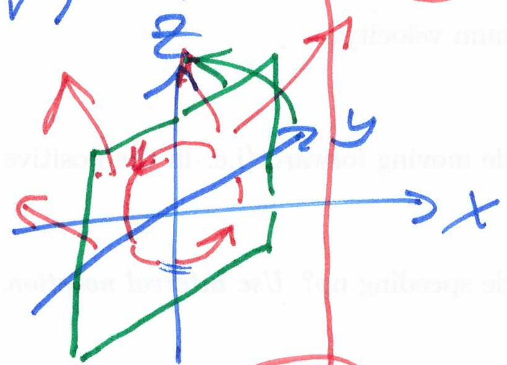
$$\operatorname{curl}\vec{F} = \left(\operatorname{curl}_{yz}\vec{F},\ -\operatorname{curl}_{xz}\vec{F},\ \operatorname{curl}_{xy}\vec{F}\right)$$

around x-axis   y-axis   z-axis

correct orientation

(p, r) (y, z)

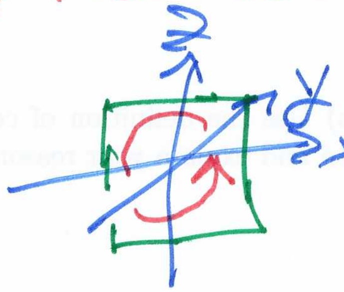
$$\operatorname{curl}_{yz}\vec{F} = \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}$$



Reverse orientation

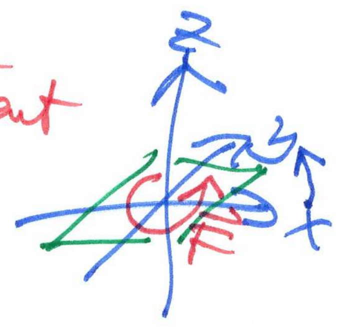
$$\operatorname{curl}_{xz}\vec{F} = \frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}$$

$(p, \cancel{q}, r)$



correct orient

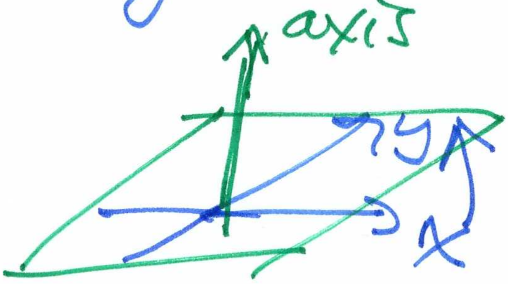
$$\operatorname{curl}_{xy}\vec{F} = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}.$$



$(p, q, \cancel{r})$ vars (x, y) $z = \text{const}$

Orientation (signs)

Righthand rule



Compute $\vec{F} = (p, q, r)$

curl $\vec{F}$ = rate of circulation rot around x y z - axes

$$\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}, -\frac{\partial r}{\partial x} + \frac{\partial p}{\partial z}, \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)$$

<u>Prop</u>: Rate of circulation around any unit vector $\vec{q}$ (in plane $\perp \vec{q}$) $|\vec{q}| = 1$

$= (\text{curl}\,\vec{F}) \cdot \vec{q}$. "directional curl"

Ex: $\vec{q} = \vec{i} = (1, 0, 0)$ gives $\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}$

Analogy: Directional derivative

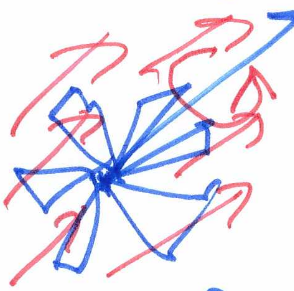
of $f(x,y,z)$. $\quad f: \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

3 coordinate rates of change

$\vec{q}$ unit vector

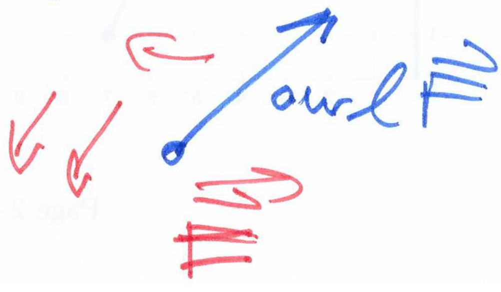
$$\frac{\partial f}{\partial \vec{q}} = \lim_{h \to 0} \frac{f(\vec{v} + h\vec{q}) - f(\vec{v})}{h}$$



$$\stackrel{!}{=} (\vec{\nabla} f) \cdot \vec{q}$$

∇f = dir of max increase

curl $\vec{F}$ = dir of maximum rotation



= primary rotation axis of $\vec{F}$ at (x,y,z)

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$$F = (p, q, r)$$

$$\text{curl } \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (p, q, r)$$

$$= \nabla \times \vec{F}$$