Math 254   3/16/2020

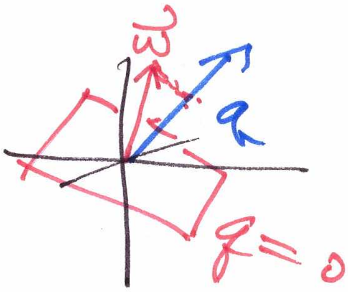
$\vec{u}, \vec{v} \in \mathbb{R}^3$ $\qquad \vec{u} \times \vec{v} = \text{vector in } \mathbb{R}^3$

vec × vec = vec

$l : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\qquad l(\vec{w}) = \det(\underbrace{\vec{u}, \vec{v}}_{\text{fixed}}, \vec{w})$ variable

linear

$$l(\vec{w}) = \underbrace{\vec{q} \cdot \vec{w}} \qquad \text{fixed } \vec{q} = \vec{u} \times \vec{w}$$



Defining eq of cross prod

$$\boxed{\det(\vec{u}, \vec{v}, \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}}$$

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \vec{i} & \vec{j} & \vec{k} \end{bmatrix}$$

cofactor exp.

$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0)$

$\vec{k} = (0,0,1)$

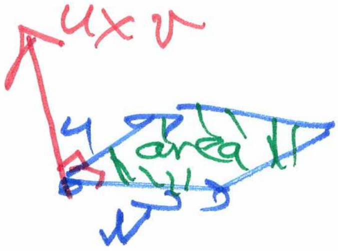
$$= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$$(\vec{u}\times\vec{v})\cdot\vec{u} = \det(\vec{u},\vec{v},\vec{u}) = 0$$

$$(\vec{u}\times\vec{v})\cdot\vec{v} = \qquad\qquad = 0$$

$\vec{u}\times\vec{v} \perp \vec{u},\vec{v}$

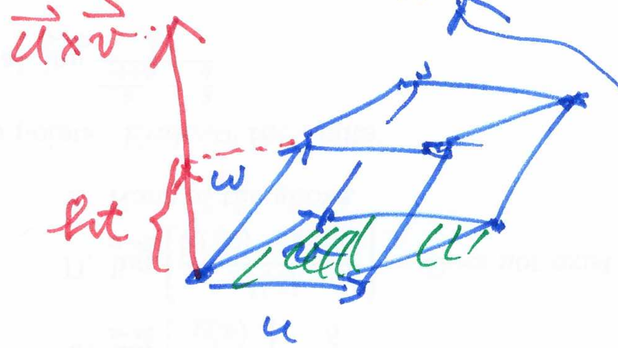
normal to $\vec{u},\vec{v}$, plane



$|u\times v| = \text{area}\ \text{Par}(u,v) = \text{area}(u,v)$

(base) · (height)

$$\text{vol BOX}(u,v,w) = \underline{\text{area}(u,v)}\ \ \vec{w}\cdot\frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}$$



$$\overset{!}{\underset{def}{=}} \det(u,v,w)$$

$$= \underline{(\vec{u}\times\vec{v})\cdot\vec{w}}$$

$$\frac{\text{area}(\vec{u},\vec{v})}{|\vec{u}\times\vec{v}|}\,\cancel{(\vec{u}\times\vec{v})}\cdot\vec{w} = \cancel{(\vec{u}\times\vec{v})}\cdot\vec{w} \quad 1$$

$$|\vec{u}\times\vec{v}| = \text{area}(\vec{u},\vec{v}) = |\vec{u}|\,|\vec{v}|\sin\theta_{\vec{u},\vec{v}}$$

exercise

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta_{\vec{u},\vec{v}}$$

Generalize 2-dim functions to 3-dim

- $\vec{c}: \mathbb{R} \to \mathbb{R}^3$ $\quad c(t) = (x(t), y(t), z(t))$

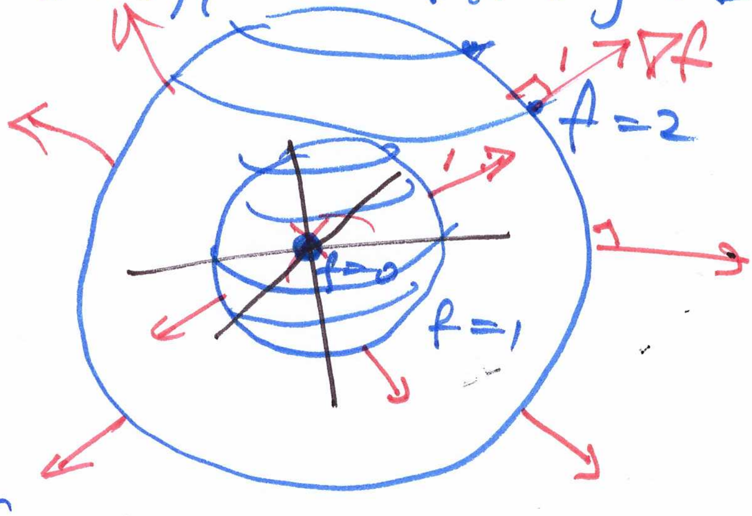
  curve in space

- $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z)$ scalar

picture? graph ~~$w = f(x,y,z)$ in $\mathbb{R}^4$~~ can't picture

Level surfaces $f(x,y,z) = c$

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ radius from $(0,0,0)$



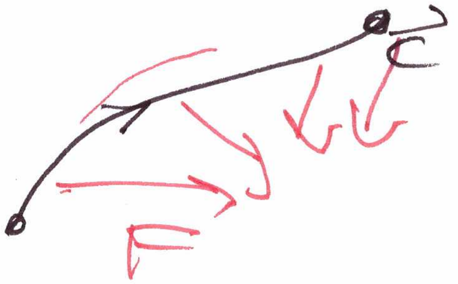
Gradient vector field

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\overbrace{(x,y,z)}^{\text{radius vector}}}{\underbrace{\sqrt{x^2+y^2+z^2}}_{\text{len of radius}}}$$

Geom meaning = dir of max increase

Line integral

$$\int_{\vec{c}} \vec{F}(\vec{c}) \cdot d\vec{c} = \text{net pull of } \vec{F} \text{ along } \vec{c}$$



$$= \int_{t=0}^{1} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

$\vec{F}$ at curve point

tangent vector.

$|\vec{c}\,'(t)| = \text{vel}$

$\vec{c}(t)$ for $t \in [0,1]$

<u>Gradient Theorem:</u>

$$f(\vec{c}(1)) - f(\vec{c}(0)) = \int \nabla f(\vec{c}) \cdot d\vec{c}$$

Total change in f along $\vec{c}$ = line integral of gradient of f along $\vec{c}$

$\vec{c}(1) = (a,b,c)$

$\vec{c}(0) = (0,0,0)$

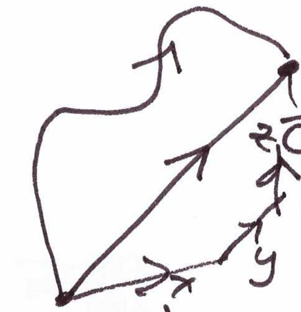
$$\underset{\uparrow \text{ find.}}{f(a,b,c)} = f(0,0,0) + \int \underbrace{\nabla f(\vec{c})}_{\text{given } \vec{F} = \nabla f} \cdot d\vec{c}$$

$\vec{F}(x,y,z) = (y, x, 1)$ picture?

Find potential fun $f(x,y,z)$

with $\nabla f = \vec{F}$

(a,b,c)



$(0,0,0)$

$\vec{c}(t) = (ta, tb, tc)$

$\vec{c}\,'(t) = (a,b,c)$ const velocity

$$f(a,b,c) = \cancelto{0}{f(0,0,0)} + \int_{t=0}^{1} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_{t=0}^{1} \vec{F}(ta, tb, tc) \cdot (a,b,c)\, dt$$

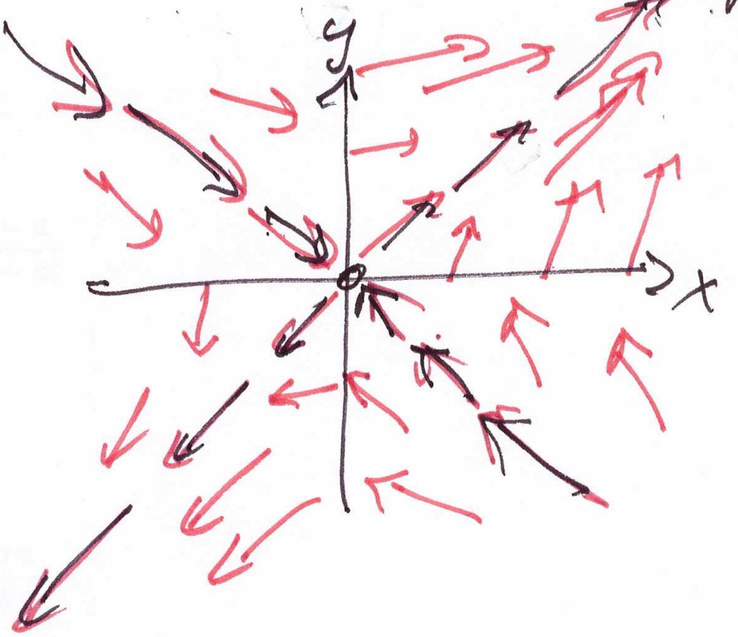
$$= \int_{t=0}^{1} (\overset{y}{tb}, \overset{x}{ta}, \overset{1}{1}) \cdot (a,b,c)\, dt$$

$$= \int_{t=0}^{1} \underline{abt + abt} + c\, dt$$

$$= \left[abt^2 + ct\right]_{t=0}^{t=1} = ab + c$$

$f(x,y,z) = xy + z$

$F(x,y,z) = (y, x, 1)$ picture



Also: z-component of 1