A combinatorial graph consists of a set of vertices or nodes $V = \{v_1, \ldots, v_k\}$ and a set of edges $e \in E$, whose elements are unordered pair of distinct vertices: $e = \{vv'\}$. We picture this as a network of nodes with certain pairs connected by wires. An unlabeled graph means an equivalence class of graphs under the symmetric group $S_k$ which permutes the vertex labels: we picture such a class by drawing all the vertices as identical balls. We make the class of (labeled or unlabeled) graphs into a graded class by taking the size function to be the number of vertices $k$.

A tree $T$ is a graph which obeys any of the equivalent definitions:

- $T$ satisfies any two (and hence all three) of the following conditions:
  - $T$ is connected (there is a path along edges between any two vertices).
  - $T$ has no cycles (embedded copies of a polygon edge-graph).
  - $T$ has $k$ vertices and $k-1$ edges.

- $T$ is minimally connected: removing any edge $e \in E$ disconnects $T-e$.

- $T$ is maximally acyclic: adding any edge $e \notin E$ produces a cycle in $T+e$.

The class $\mathcal{T}$ of unlabeled trees is quite difficult to enumerate, because there is no well-defined first or last vertex to delete to get a recurrence. However, let $\mathcal{R}$ be the class of rooted unlabeled trees, having an extra piece of data: any one of the vertices is distinguished and called the root. Letting $r_k$ be the number of rooted, unlabelled trees with $k$ vertices, we have: $r_0 = 0$, $r_1 = r_2 = 1$, $r_3 = 2$, $r_4 = 4$. For example, for $n = 4$, the tree can be either: a linear path rooted at an end or an internal vertex; or a star rooted at the center or at a leaf; giving $r_4 = 4$ distinct choices.

Deleting the root, and taking its neighbors as the roots of a set of new trees (some of which may be repeated), gives the combinatorial specification:

$$\mathcal{R} = [1] \times \text{MSet}(\mathcal{R}).$$

Note this is consistent with $r_0 = 0$. This translates into the following equation involving the ordinary generating function $R(x) = \sum_{n \geq 1} r_k x^k$:

$$R(x) = x \prod_{i \geq 1} \frac{1}{(1-x^i)^{r_i}}.$$

We will apply logarithmic differentiation to obtain an amazing recurrence for $r_k$. Writing out the equation as:

$$\sum_{k \geq 0} r_{k+1} x^k = \prod_{i \geq 1} (1-x^i)^{-r_i},$$

we apply the operation $x \frac{d}{dx} \log$ to both sides. On the left side, the identity $x \frac{d}{dx} \log f(x) = \frac{x f'(x)}{f(x)}$ implies:

$$x \frac{d}{dx} \log \sum_{k \geq 0} r_{k+1} x^k = \frac{\sum_{k \geq 1} k r_{k+1} x^k}{\sum_{i \geq 0} r_{k+1} x^k}.$$
On the right side, we use $\log(ab) = \log a + \log b$ and $\log(a^b) = b \log a$ to get:

$$x \frac{d}{dx} \log \prod_{i \geq 1} (1 - x^i)^{-r_i} = \sum_{i \geq 1} -r_i x \frac{d}{dx} \log(1 - x^i)$$

$$= \sum_{i \geq 1} r_i \frac{ix^i}{1 - x^i} = \sum_{i \geq 1} \sum_{m \geq 1} i r_i x^{im} = \sum_{j \geq 1} (\sum_{i \mid j} i r_i) x^j .$$

where in the last equality we substitute $j = im$, and $i \mid j$ means $i$ divides $j$.

Now equating the two sides, clearing the denominator, and collecting $x^k$ terms:

$$\sum_{k \geq 1} kr_{k+1} x^k = \sum_{j \geq 1} (\sum_{i \mid j} i r_i) x^j \cdot \sum_{\ell \geq 0} r_{\ell+1} x^\ell = \sum_{k \geq 1} (\sum_{j \geq 1} \sum_{i \mid j} i r_i r_{k-j+1}) x^k$$

where in the second equality we substitute $k = j + \ell$, so that $\ell + 1 = k - j + 1$. We conclude:

$$r_{k+1} = \frac{1}{k} \sum_{j \geq 1} \sum_{i \mid j} i r_i r_{k-j+1} ,$$

where the right side involves only $r_1, \ldots, r_k$. This recurrence has no combinatorial explanation, but it is fairly efficient computationally.

**Example:** To compute $r_5$, we sum over $j = 1, 2, 3, 4$ and $i$ running over all divisors of $j$: that is, $(i, j) = (1, 1), (1, 2), (2, 2), (1, 3), (3, 3), (1, 4), (2, 4), (4, 4)$, so that:

$$r_5 = \frac{1}{4} (r_1 r_4 + r_1 r_3 + 2r_2 r_3 + r_1 r_2 + 3r_3 r_2 + r_1 r_1 + 2r_2 r_1 + 4r_4 r_1)$$

$$= \frac{1}{4} (4 + 2 + 4 + 1 + 6 + 1 + 2 + 16) = 9 .$$

In fact, a tree with 5 vertices can be a path with 3 distinct locations for the root, a Y-shape with 4 locations, or a star with 2 locations, which also makes $r_5 = 3 + 4 + 2 = 9$. 