Math 481 # Midterm Exam Solutions **Mar 1, 2021**

Instructions:

- Write your name on each page.
- Show your work and explain your reasoning. Explicitly refer to principles, problems, and methods.
- You MAY use your own notes and our course materials: board notes, Main Page, HHM textbook.
- You MAY NOT use other internet references, or personal help from anyone else.
- Upload to Gradescope as for quizzes, before 10:10am.

Setup for problems 1-4: A town elects a mayor, a treasurer, and three council members from among 100 citizens (Citizen #1, . . . , Citizen #100).

1a. (10pt) How many possible election results if 5 different citizens are elected to the 5 positions?
EXAMPLE: mayor #42, treasurer #10, council #18, 57, 98.
Simplify your answer in terms of arithmetic operations only (no choose numbers, etc).

SOLUTION: Choose 1 of 100 citizens as mayor, then 1 of the remaining 99 as treasurer, then a set of 3 of the remaining 98 as council members.
ANSWER:

$$100 \cdot 99 \cdot \binom{98}{3} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{3 \cdot 2}$$

This is also the multinomial coefficient $\binom{100}{1,1,3,95}$, the number of ways to split the citizens into subsets of size 1,1,3,95.

1b. (10pt) How many possible results if one citizen can win more than one position (but only one council seat)? EXAMPLE: mayor #10, treasurer #10, council #10, 57, 98.
Simplify your answer in terms of arithmetic operations only (no choose numbers, etc).

SOLUTION: Same, but no need to reduce choices in each step.
ANSWER:

$$100 \cdot 100 \cdot \binom{100}{3} = \frac{100^3 \cdot 99 \cdot 98}{3 \cdot 2}$$

2. (15pt) There are 4 candidates for mayor (Q,R,S,T). How many ways can they split 100 votes?
EXAMPLE: Q gets 15 votes, R gets 45 votes, S gets 20 votes, T gets 20 votes.
Simplify your answer in terms of arithmetic operations only (no choose numbers, etc).

SOLUTION: The problem counts lists of whole numbers (q, r, s, t) with $q + r + s + t = 100$.
The Multiplicity Transform takes these to multisets of 100 objects chosen from 4 types.
ANSWER:

$$\left(\!\!\binom{4}{100}\!\!\right) = \binom{103}{100} = \binom{103}{3} = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2}$$

3. (25pt) How many ways can 4 candidates split 100 votes, if none gets 40 or more votes? Use PIE.

a. Define the sets of all results A and bad results $B_1, B_2, \ldots$. (Define sets, not numbers.)

SOLUTION: $A = \{\text{all ways for 4 candidates to split 100 votes}\}$
$B_1 = \{\text{ways with candidate Q getting} \geq 40 \text{ votes}\}$, and similarly for B_2 and R; B_3 and S; B_4 and T.

b. Write the general PIE formula in terms of $|A|$, $|B_i|$, etc. (Write a complete equation.)

SOLUTION:

$$|A - (B_1 \cup B_2 \cup B_3 \cup B_4)| \;=\; |A| - \sum_i |B_i| + \sum_{i<j} |B_i \cap B_j| - \sum_{i<j<k} |B_i \cap B_j \cap B_k| + |B_1 \cap B_2 \cap B_3 \cap B_4|$$

c. Solve the problem by evaluating the terms of the PIE formula.
Your answer may contain choose numbers, and other symbols.

SOLUTION: To count B_i, start with 40 votes for one candidate, then split the remaining 60 among the 4 candidates in $\left(\!\binom{4}{60}\!\right)$ ways. Similarly $|B_i \cap B_j| = \left(\!\binom{4}{20}\!\right)$, and the higher intersections are empty.

ANSWER:

$$\left(\!\!\binom{4}{100}\!\!\right) - \binom{4}{1}\left(\!\!\binom{4}{60}\!\!\right) + \binom{4}{2}\left(\!\!\binom{4}{20}\!\!\right) - 0 + 0$$

4. (25pt) For $k \geq 0$, let a_k be the number of ways 4 candidates can split k votes with none getting 40 or more votes. Let $f(x) = \sum_{k \geq 0} a_k x^k$ be the generating function.
Problem: Write an algorithm for distributing votes, and translate this into a formula for $f(x)$. Simplify $f(x)$ as much as possible, into a short formula.
Note: Do NOT go on to Step 2. Do NOT find a formula for a_k.

SOLUTION: Algorithm:

$$\left(\begin{array}{c}\text{Vote tallies for 4 candidates}\\ \text{with} < 40 \text{ votes for each}\end{array}\right) \iff \begin{array}{c}(0Q \text{ or } 1Q \text{ or } \cdots \text{ or } 39Q)\\ \text{and } (0R \text{ or } 1R \text{ or } \cdots \text{ or } 39R)\\ \text{and } (0S \text{ or } 1S \text{ or } \cdots \text{ or } 39S)\\ \text{and } (0T \text{ or } 1T \text{ or } \cdots \text{ or } 39T)\end{array}$$

Translating to algebra, a choice of m votes for a candidate is translated to x^m.

$$f(x) = (x^0 + x^1 + \cdots + x^{39}) \cdots (x^0 + x^1 + \cdot + x^{39}) = (1 + x + \cdots + x^{39})^4 = \left(\frac{1 - x^{40}}{1 - x}\right)^4$$

Here we used the sum of a finite geometric series.

5. (15pt) Find the Taylor series of the function

$$g(x) \;=\; \frac{1+2x}{(1+x)^3} \;=\; \sum_{k\geq 0} b_k x^k\,.$$

Compute a formula for b_k. Simplify in terms of arithmetic operations only.
Note: If you cannot get a general formula, compute at least the first few $b_0, b_1, \ldots$.

SOLUTION: Using the Negative Binomial Series $\frac{1}{(1-z)^n} = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) z^k$, we get

$$\begin{aligned} g(x) &= (1+2x)\frac{1}{(1-(-x))^3} \\ &= 1\textstyle\sum_{k\geq 0} \left(\!\binom{3}{k}\!\right)(-x)^k + 2x\sum_{k\geq 0} \left(\!\binom{3}{k}\!\right)(-x)^k \\ &= \textstyle\sum_{k\geq 0}(-1)^k \left(\!\binom{3}{k}\!\right) x^k + \sum_{k\geq 0}(-1)^k 2 \left(\!\binom{3}{k}\!\right) x^{k+1} \\ &= \textstyle\sum_{k\geq 0}(-1)^k \left(\!\binom{3}{k}\!\right) x^k + \sum_{k\geq 1}(-1)^{k-1} 2 \left(\!\binom{3}{k-1}\!\right) x^k \end{aligned}$$

Thus for $k \geq 1$, the coefficient of x^k is:

$$\begin{aligned} b_k &= (-1)^k \left(\!\binom{3}{k}\!\right) + (-1)^{k-1} 2 \left(\!\binom{3}{k-1}\!\right) \\ &= (-1)^k \left(\binom{k+2}{2} - 2\binom{k+1}{2}\right) \\ &= (-1)^k \left(\tfrac{(k+2)(k+1)}{2} - 2\tfrac{(k+1)k}{2}\right) \\ &= \tfrac{(-1)^k}{2}(k+1)(k+2-2k) \\ &= \tfrac{(-1)^{k+1}}{2}(k+1)(k-2) \end{aligned}$$

Another way to expand comes from the higher partial fraction decomposition:

$$\frac{1+2x}{(1+x)^3} \;=\; \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{(1+x)^3},$$

which we can solve to get $A=0$, $B=-2$, $C=1$, and expand with the Negative Binomial Series:

$$b_k \;=\; (-1)^k \left(-2\left(\!\binom{2}{k}\!\right) + \left(\!\binom{3}{k}\!\right)\right).$$

A way to compute the first few coefficients:

$$\begin{aligned} \frac{1+2x}{(1+x)^3} &= (1+2x)(1-x+x^2-x^3+\cdots)^3 \\ &= (1+2x)(1-3x+3x^2+3xx-3x^3-6x^2x-xxx+\cdots) \\ &= (1-3x+6x^2-10x^3+\cdots)+(2x-6x^2+12x^3+\cdots) \\ &= 1-x+0x^2+2x^3+\cdots \end{aligned}$$