**Math 133** Partial Fractions  **Stewart §7.4**

**Integrating basic rational functions.** For a function \( f(x) \), we have examined several algebraic methods\(^*\) for finding its indefinite integral (antiderivative) \( F(x) = \int f(x) \, dx \), which allows us to compute definite integrals \( \int_a^b f(x) \, dx = F(b) - F(a) \) by the Second Fundamental Theorem.

In this section, we will learn a special technique to integrate any \textit{rational function}, meaning a quotient of two polynomials:

\[
 f(x) = \frac{g(x)}{h(x)} = \frac{a_m x^n + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0},
\]

where \( a_i, b_j \) are constant coefficients. We call the largest powers \( m \) and \( n \) the \textit{degrees} of the polynomials \( g(x) \) and \( h(x) \), assuming that the highest coefficients \( a_m, b_n \neq 0 \).

We have several basic rational functions whose integrals we already know:

(i) \( \int a_m x^n + \cdots + a_1 x + a_0 \, dx = \frac{a_m}{m+1} x^{m+1} + \cdots + \frac{a_1}{2} x^2 + a_0 x + C. \)

(ii) \( \int \frac{1}{x-a} \, dx = \ln|x-a| + C. \)

(iii) \( \int \frac{1}{(x-a)^n} \, dx = -\frac{1}{(n-1)(x-a)^{n-1}} \) for \( n \geq 2. \)

(iv) \( \int \frac{x}{x^2+a} \, dx = \frac{1}{2} \int \frac{1}{x^2+a} \cdot 2x \, dx = \frac{1}{2} \ln|x^2+a| + C \)

(v) \( \int \frac{1}{x^2+a} \, dx = \frac{1}{\sqrt{a}} \int \frac{1}{\left(\frac{x}{\sqrt{a}}\right)^2+1} \cdot \frac{1}{\sqrt{a}} \, dx = \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + C, \) for \( a > 0. \)

(vi) \( \int \frac{1}{(x^2+1)^2} \, dx. \) Letting \( x = \tan(\theta), \ x^2+1 = \sec^2(\theta), \ dx = \sec^2(\theta) \, d\theta: \)

\[
\int \frac{1}{(x^2+1)^2} \, dx = \int \frac{1}{\sec^4(\theta)} \sec^2(\theta) \, d\theta = \int \cos^2(\theta) \, d\theta
\]

\[
= \frac{1}{2} \left( \theta + \sin(\theta) \cos(\theta) \right) + C = \frac{1}{2} \left( \arctan(x) + \frac{x}{x^2+1} \right) + C.
\]

We used: \( \int \cos^2(\theta) \, d\theta = \int \frac{1}{2} + \frac{1}{2} \cos(2\theta) \, d\theta = \frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) = \frac{1}{2} \theta + \frac{1}{2} \sin(\theta) \cos(\theta), \)

and (as in §6.6) \( \sin(\theta) = \frac{x}{\sqrt{x^2+1}}, \ \cos(\theta) = \frac{1}{\sqrt{x^2+1}}. \)

**Quadratic denominator.** With the above basic integrals, we can integrate any rational function with numerator of degree at most 1 and denominator of degree at most 2:

\[
\int \frac{px+q}{ax^2+bx+c} \, dx.
\]

There are two different cases, depending on the sign of the \textit{discriminant} \( d = b^2 - 4ac. \)

\(^*\)Substitution §4.5, Integration by Parts §7.1, Products of Trig Functions §7.2, Reverse Trig Substitution §7.3

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EXAMPLE: Here is how to handle the case where \( d = b^2 - 4ac > 0 \), such as:

\[
\int \frac{x + 1}{x^2 + x - 2} \, dx,
\]

where \( d = 1^2 - 4(1)(-2) = 9 \). By the Quadratic Formula, the denominator has two real roots \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 1, -2 \), which are the vertical asymptotes of our function:

We split our function into a sum of simple parts, each having just one vertical asymptote:

\[
\frac{x + 1}{x^2 + x - 2} = \frac{x + 1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.
\]

This is called the partial fraction expansion of our rational function. For any constants \( A, B \), the graph of the right-hand function will have the same asymptotes as our original function, but we can actually find constants which make the two exactly equal. Clearing denominators, we want \( A, B \) such that:

\[
x + 1 = A(x+2) + B(x-1) \quad \text{for all } x.
\]

Setting \( x = 1 \) gives \( 1 + 1 = A(1+2) + B(0) \), so \( A = \frac{2}{3} \); and setting \( x = -2 \) gives \( -2 + 1 = A(0) + B(-2-1) \), so \( B = \frac{1}{3} \). Now we can use the basic integral (ii) above:

\[
\int \frac{x + 1}{x^2 + x - 2} \, dx = \int \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2} \, dx = \frac{2}{3} \ln|x-1| + \frac{1}{3} \ln|x+2| + C.
\]

EXAMPLE: The other case is when \( d = b^2 - 4ac < 0 \), such as:

\[
\int \frac{x + 1}{x^2 + x + 1} \, dx,
\]

for which \( d = 1^2 - 4(1)(1) = -3 \). In this case, the denominator has no real-number zeroes: \( x^2 + x + 1 > 0 \), and it cannot be factored; hence the graph of \( \frac{x + 1}{x^2 + x + 1} \) has no vertical asymptotes. Our strategy is to reduce the integral to the basic integrals (iii) and (iv) above.

The first step is to complete the square in the denominator and force it into the form \((x+p)^2 + q\), the same process that produces the Quadratic Formula:

\[
x^2 + x + 1 = x^2 + 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.
\]
Thus, letting $u = x + \frac{1}{2}$, $du = dx$:
\[
\int \frac{x + 1}{x^2 + x + 1} \, dx = \int \frac{(x+\frac{1}{2}) - \frac{1}{2} + \frac{1}{2}}{(x+\frac{1}{2})^2 + \frac{3}{4}} \, dx
\]
\[
= \int \frac{u}{u^2 + \frac{3}{4}} \, du + \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} \, du
\]
\[
= \frac{1}{2} \ln|u^2+\frac{3}{4}| + \frac{1}{2\sqrt{3/4}} \arctan\left(\frac{u}{\sqrt{3/4}}\right) + C
\]
\[
= \frac{1}{2} \ln|x^2+x+1| + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.
\]
In fact, the two terms in our answer correspond to splitting the original function (blue) into a graph with reflection symmetry across the line $x = -\frac{1}{2}$ (green), and a graph with 180° rotation symmetry around the point $(-\frac{1}{2}, 0)$ (red):

**Example:** One more case: if the numerator has degree greater than or equal to the denominator, for example:
\[
\int \frac{x^4 + 2x + 3}{x^2 + x - 2} \, dx.
\]
Then $y = 0$ is no longer a horizontal asymptote. Instead, the behavior of the function as $x \to \pm\infty$ is controlled by a polynomial curve obtained by polynomial long division.

\[
x^2 - x + 3 \quad \text{rem} \quad -3x + 9
\]
\[
x^2 + x - 2 \quad \left(\begin{array}{c}
\frac{x^4}{x^4 + x^3 - 2x^2} + 2x + 3 \\
-(x^4 + x^3 - 2x^2)
\end{array}\right)
\]
\[
\quad \quad = -x^3 + 2x^2 + 2x + 3
\]
\[
\quad \quad \quad \quad \quad \quad \quad = -(x^3 - x + 2x)
\]
\[
\quad \quad \quad \quad \quad \quad \quad = \frac{3x^2}{3x^2 + 3x - 6} + 3
\]
\[
\quad \quad \quad \quad \quad \quad \quad = \frac{-3x + 9}{3x^2 + 3x - 6}
\]
\[
\quad \quad \quad \quad \quad \quad \quad = \frac{-3x + 9}{3x^2 + 3x - 6}
\]

Thus $x^4 + 2x + 3 = (x^2-x+3)\cdot(x^2+x-2) + (-3x+9)$, and:
\[
\frac{x^4 + 2x + 3}{x^2 + x - 2} = (x^2-x+3) + \frac{-3x + 9}{x^2 + x + 1} = (x^2-x+3) + \frac{2}{x - 1} - \frac{5}{x + 2}
\]
The last equality is a partial fraction expansion similar to our first example above. Now:
\[
\int \frac{x^4 + 2x + 3}{x^2 + x - 2} \, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + 2 \ln|x-1| - 5 \ln|x+2| + C.
\]
Hence, according to (i)–(vi):

We solve this as:

\[ x \]

Since this is an equality of polynomial functions, the coefficients of \( k \) for all \( x-r \):

We need to find the six constants \( A \) in a decomposition of \( g \) into a sum of terms of the following forms:

- A polynomial \( q(x) \), which is the quotient in the long division \( g(x) \div h(x) = q(x) \) with remainder \( r(x) \).

- For each linear factor \( x-r \) of the denominator \( h(x) \), suppose \( (x-r)^n \) is the highest power which divides \( h(x) \). Then we add a sum of \( n \) terms:

\[
\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_n}{(x-r)^n}.
\]

- For each irreducible quadratic factor \( ax^2 + bx + c \) of \( h(x) \), suppose \( (ax^2 + bx + c)^n \) is the highest power which divides \( h(x) \). Then we add a sum of \( n \) terms:

\[
\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n}.
\]

Setting \( f(x) = g(x)/h(x) \) equal to the sum of all the terms above, we clear the denominators and solve for all the unknown constants in the numerators as we did for \( A, B \) in our first example above. Once this is done, we can integrate using (i)–(vi) and the above examples.\(^1\)

**Example:** We find the partial fraction expansion of:

\[ f(x) = \frac{1}{x^2(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1 x + C_1}{x^2 + 1} + \frac{B_2 x + C_2}{(x^2 + 1)^2}. \]

We need to find the six constants \( A_1, A_2, B_1, B_2, C_1, C_2 \) which make the above equation valid. Clearing denominators gives:

\[
1 = A_1 x(x^2+1)^2 + A_2(x^2+1)^2 + (B_1 x + C_1) x^2 (x^2+1) + (B_2 x + C_2) x^2
\]

\[
= (A_1 + B_1) x^5 + (A_2 + C_1) x^4 + (2A_1 + B_1 + B_2) x^3 + (2A_2 + C_1 + C_2) x^2 + A_1 x + A_2
\]

Since this is an equality of polynomial functions, the coefficients of \( x^k \) must be the same for all \( k \):

\[
A_1 + B_1 = 0
\]

\[
A_2 + C_1 = 0
\]

\[
2A_1 + B_1 + B_2 = 0
\]

\[
2A_2 + C_1 + C_2 = 0
\]

\[
A_1 = 0, \quad A_2 = 1
\]

\[
B_1 = -A_1 = 0, \quad C_1 = -A_2 = -1,
\]

\[
B_2 = -2A_1 - B_1 = 0, \quad C_2 = -2A_2 - C_1 = -1.
\]

We solve this as:

\[
A_1 = 0, \quad A_2 = 1, \quad B_1 = -A_1 = 0, \quad C_1 = -A_2 = -1,
\]

\[
B_2 = -2A_1 - B_1 = 0, \quad C_2 = -2A_2 - C_1 = -1.
\]

Hence, according to (i)–(vi):

\[
\int \frac{1}{x^2(x^2+1)^2} \, dx = \int \frac{1}{x^2} - \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \, dx
\]

\[
= \frac{1}{x} - \arctan(x) - \frac{1}{2} \left( \arctan(x) + \frac{x}{x^2 + 1} \right).
\]

\(^1\)For \( \int \frac{1}{x^2+1} \, dx \) we need even more elaborate contortions. The truth is this only becomes manageable if we use imaginary number factorizations like \( x^2 + 1 = (x - \sqrt{-1})(x + \sqrt{-1}) \), avoiding quadratic denominators entirely.
Trig integrals again. In §7.2–7.3, we reduced trig integrals by Substitution to rational function integrals, which we can now find by Partial Fractions. For example:

\[
\int \sec(x) \, dx = \int \frac{1}{\cos^2(x)} \cdot \cos(x) \, dx = \int \frac{1}{1 - \sin^2(x)} \cdot \cos(x) \, dx
\]

\[
= \int \frac{1}{1 - u^2} \, du = \int \frac{1}{1 - u} + \frac{1}{1 - u} \, du
\]

\[
= \frac{1}{2} \ln(1+u) - \frac{1}{2} \ln(1-u) = \ln \sqrt{\frac{1+u}{1-u}} = \ln \sqrt{\frac{1+\sin(x)}{1-\sin(x)}}.
\]

Challenge problem: Show by identities that this is equal to our previous answer \(\int \sec(x) \, dx = \ln|\tan(x) + \sec(x)|\) given in §7.2. Also: try this method on \(\int \sec^3(x) \, dx\).