### Example 1.
Let \( f(x) = 2x^2 + 4x \). Find the slope of the secant line between \((4, f(4))\) and \((4 + h, f(4 + h))\).

- **Solution.**
  \[
m = \frac{f(4 + h) - f(4)}{(4 + h) - 4} = \frac{2(4 + h)^2 + 4(4 + h) - (2(4)^2 + 4(4))}{h} = \frac{2(16 + 8h + h^2) + 16 + 4h - (32 + 16)}{h} = \frac{16h + 2h^2 + 4h}{h} = 20 + 2h
  \]

### Example 2.
Evaluate the following limit. \( \lim_{x \to 5} \frac{x^2 + 1}{x^2 - 10x + 25} \).

- **Solution.**
  \[
  \lim_{x \to 5} \frac{x^2 + 1}{(x - 5)^2} = \lim_{x \to 5} \frac{x^2 + 1}{(x - 5)^2} \to\infty
  \]
  So, \( \lim_{x \to 5} \frac{x^2 + 1}{(x - 5)^2} \to\infty \) \( \lim_{x \to 5} \frac{x^2 + 1}{(x - 5)^2} \to\infty \)

### Example 3.
Evaluate the following limit. \( \lim_{t \to 1^+} \frac{\sqrt{tt}(t - 1)}{|t - 1|} \).

- **Solution.**
  For \( t \geq 1 \), \( |t - 1| = t - 1 \)
  So,
  \[
  \lim_{t \to 1^+} \frac{\sqrt{tt}(t - 1)}{|t - 1|} = \lim_{t \to 1^+} \frac{\sqrt{tt} = \sqrt{t}}
  \]

### Example 4.
Prove the following limit statement using the definition of the limit. \( \lim_{x \to 3} x^2 - 6x + 18 = 9 \).
That is for \( \epsilon > 0 \), find the largest \( \delta \) for the formal definition of the limit to hold.

- **Solution.**
  \[
  \lim_{x \to a} f(x) = L, \quad \lim_{x \to 3} x^2 - 6x + 19 = 9
  0 < |x - a| < \delta, \quad 0 < |x - 3| < \delta
  |f(x) - L| < \epsilon, \quad |x^2 - 6x + 18 - 9| < \epsilon,
  |x^2 - 6x + 9| < \epsilon, \quad |(x - 3)|^2 < \epsilon, \quad |x - 3| < \sqrt{\epsilon}
  So, \( \delta = \sqrt{\epsilon} \)
  \]

### Example 5.
Let \( f(x) = \frac{2x + 5}{\cos 9x} \) where \( x \in \left(-\frac{\pi}{9}, \frac{\pi}{9}\right) \), write down the interval where the function is continuous.

- **Solution.**
  \[
  \cos(9x) = 0
  9x = \pm \frac{\pi}{2} + \pi k, \quad x = \pm \frac{\pi}{18} + \frac{\pi}{2} k
  9x = -\frac{\pi}{2} - \pi k, \quad x = -\frac{\pi}{18} - \frac{\pi}{2} k
  So, \( x \neq -\frac{\pi}{18}, -\frac{\pi}{9} \)
  \( f(x) \) continuous on \( \left[-\frac{\pi}{9}, -\frac{\pi}{18}\right] \cup \left(-\frac{\pi}{18}, \frac{\pi}{18}\right] \cup \left(\frac{\pi}{18}, \frac{\pi}{9}\right] \)
Example 6. Let \( f(x) = \frac{9}{x+8} \). Use this function to find \( \frac{f(z) - f(x)}{z-x} \). Then, use this to find the derivative of \( f(x) \) by taking the proper limit.

- **Solution.**

\[
\frac{f(z) - f(x)}{z-x} = \frac{\frac{9}{z+8} - \frac{9}{x+8}}{z-x} = \frac{9}{(z+8)(x+8)} \frac{x+8-(z+8)}{(z+8)(x+8)} = -9
\]

\[
\lim_{z \to x} \frac{f(z) - f(x)}{z-x} = -9 \div (x+8)^2
\]

Example 7. Use the definition of the derivative to find \( r'(t) \) given \( r(t) = 3t^\frac{1}{2} + 4 \).

- **Solution.**

\[
r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} = \lim_{h \to 0} \frac{3(t+h)^\frac{1}{2} + 4 - (3t + 4)}{h}
\]

\[
= \lim_{h \to 0} \frac{3\sqrt{t+h} - 3\sqrt{t}}{h} = \lim_{h \to 0} \frac{3 - \frac{h}{\sqrt{t+h} + \sqrt{t}}}{h}
\]

Example 8. Find the derivative of the given function: \( f(x) = (x^\frac{1}{2} + 13)(6\sqrt{x} + 5) \).

- **Solution.**

\[
f'(x) = \frac{1}{2} x ^{-\frac{1}{2}} (6\sqrt{x} + 5) + (x^\frac{1}{2} + 13) (3x^{-\frac{1}{2}})
\]

Example 9. Find the equation of the tangent to the curve \( f(x) = 3 \sec(x) - 6 \) at \( x = \frac{\pi}{6} \).

- **Solution.**

\[
f'(x) = 3 \tan(x) \sec(x), \quad f'(\frac{\pi}{6}) = 3 \tan(\frac{\pi}{6}) \sec(\frac{\pi}{6}) = 3(\sqrt{3})(\frac{2\sqrt{3}}{3}) = 2
\]

\[
f(\frac{\pi}{6}) = 3 \sec(\frac{\pi}{6}) - 6 = 3(\frac{2\sqrt{3}}{3}) - 6 = 2\sqrt{3} - 6
\]

\[
y - 2\sqrt{3} + 6 = 2(x - \frac{\pi}{6})
\]

\[
y = 2(x - \frac{\pi}{6}) + 2\sqrt{3} - 6
\]

Example 10. Given \( f(x) = \sin\left(\sqrt{\frac{3}{1+x}}\right) \). Find the derivative of \( f(x) \).

- **Solution.**

\[
f'(x) = \cos\left(\sqrt{\frac{3}{1+x}}\right) \frac{1}{2} \left(3\frac{1}{1+x}\right)^{-\frac{1}{2}} \left(-\frac{3}{(1+x)^2}\right)
\]

Example 11. An object has height described by the equation \( s(t) = 12t - 3gt^2 \). Where \( g \) is some constant. If the max value of \( s(t) \) occurs at 24 seconds, then what is the value of \( g \)?

- **Solution.**

\[
s'(t) = 12 - 6gt, \quad s'(24) = 0 \quad s'(24) = 12 - 6g(24) = 0, \quad 12 = 6(24)g, \quad g = \frac{1}{12}
\]
Example 12. Write the equation of the tangent line to the curve \( y^2 + 18x = x^2y + 40 \) at \( x = 3 \).

- **Solution.**
  Find coordinate corresponding to \( x = 3 \)
  \[ y^2 + 18(3) = (3)^2y + 40, \quad y^2 - 9y + 14 = 0, \quad y = 2, \ 7 \]
  After taking the derivative with respect to \( x \),
  \[ 2yy' + 18 = 2xy + x^2y', \]
  using point \((3, 2)\),
  \[ 4y' + 18 = 12 + 9y', \quad 5y' = 6, \quad y' = \frac{6}{5} \]
  \[ y - 2 = \frac{6}{5}(x - 3) = y = \frac{6}{5}(x - 3) + 2 \]
  Using point \((3, 7)\),
  \[ 14y' + 18 = 42 + 9y', \quad 5y' = 24, \quad y' = \frac{24}{5} \]
  \[ y - 7 = \frac{24}{5}(x - 3) = y = \frac{24}{5}(x - 3) + 7 \]

Example 13. A boat is pulled into a dock by a rope that is attached to the bow of the boat by a pulley on the dock that is 4 meters higher than the bow of the boat. If the rope is being pulled in at a rate of 3 meters per second, what is the speed that the boat is coming into the dock when it is 10 meters away?

- **Solution.**
  \[ A^2 + d^2 = r^2, r:=\text{rope}, \ d:=\text{horizontal distance to dock}, \ 4 \text{ comes from height of pulley.} \]
  \[ 16 + 10^2 = r^2, \quad r = \sqrt{116} \]
  \[ 2dd' = 2rr', \quad d' = \frac{3\sqrt{116}}{10} \]

Example 14. The radius of a sphere was measured to be 23 centimeters with maximum error of \( \frac{1}{4} \) centimeter. Use a differential to estimate the maximum error in the surface area?

- **Solution.**
  \[ \Delta A = A'(r)(r + \Delta r - r) \]
  \[ A = 4\pi r^2, \quad A' = 8\pi r \quad A'(23) = 8\pi(23) = 184\pi \]
  \[ \Delta A = 184\pi(\frac{1}{4}) = \Delta A = 46 \text{cm}^2 \]

Example 15. Let \( g(x) = 3x^2 \). On the interval \([-216, 8]\) find the absolute max and mins of the curve.

- **Solution.**
  \( g' = 2x^{-\frac{1}{3}} \) only critical point \( x = 0 \)
  \[ g(-216) = 3(-216)^{\frac{2}{3}} = 3(36) = 108 \]
  \[ g(0) = 3(0)^{\frac{2}{3}} = 0 \]
  \[ g(8) = 3(8)^{\frac{2}{3}} = 12 \]
  Abs. max=108, abs. min=0

Example 16. Determine if the mean value theorem applies to the function \( f(x) = 3\sqrt{x} - 2x \) on the interval \([9,25]\). If it does, then find the mean value and where it occurs.

- **Solution.**
  \( f(x) \) is continuous on \([9,25]\) and differentiable on \((9,25)\)
  \[ f'(c) = \frac{f(25) - f(9)}{25 - 9} = \frac{3\sqrt{25} - (3\sqrt{5} - 2(9))}{16} = \frac{15 - 50 - 9 + 18}{16} = \frac{26}{16} \]
  \[ f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - 2 \]
  \[ \frac{3}{2}c^{-\frac{1}{2}} - 2 = -\frac{26}{16}, \quad \frac{3}{2}c^{-\frac{1}{2}} = \frac{3}{8}, \quad c^{-\frac{1}{2}} = \frac{1}{4}, \quad c=16 \]
Example 17. Find where the curve \( h(x) = 5x^3 - 13x + 25 \) is increasing/decreasing and where it is concave up/down.

- **Solution.**
  \[
  h'(x) = 15x^2 - 13, \quad 0 = 15x^2 - 13, \quad x = \pm \sqrt{\frac{13}{15}} \quad h'(-1) = 2, \quad h'(0) = -13, \quad h'(1) = 2
  \]
  
  \[
  \text{Incr:} \left( -\infty, -\frac{\sqrt{13}}{15} \right), \left( \frac{\sqrt{13}}{15}, \infty \right) \quad \text{Decr:} \left( -\frac{\sqrt{13}}{15}, \frac{\sqrt{13}}{15} \right)
  \]

  \[
  f''(x) = 30x, \quad f''(x) = 0, \quad x = 0
  \]

  \[
  f''(-1) = -30, \quad f''(1) = 30
  \]

  \[
  \text{Concave Down:} \left( -\frac{\sqrt{13}}{15}, \frac{\sqrt{13}}{15} \right), \quad \text{Concave Up:} \left( 0, \infty \right)
  \]

Example 18. Find the following limit: \( \lim_{x \to \infty} (\sqrt{5x + 4} - x) \)

- **Solution.**
  \[
  \lim_{x \to \infty} (\sqrt{5x + 4} - x) = \lim_{x \to \infty} \frac{5x + 4 - x^2}{\sqrt{5x + 4} + x} = \lim_{x \to \infty} \frac{5 + \frac{4}{x} - x}{\sqrt{5 + \frac{4}{x} + 1} - x} \to -\infty
  \]

Example 19. Let \( k(x) = \frac{x + 2}{10x + 2} \). Find the following: (a) domain, (b) \( x \)-intercepts, (c) \( y \)-intercepts, (d) horizontal asymptotes, (e) vertical asymptotes, (f) increasing/decreasing, (g) concave up/concave down, (h) local max and mins, and (i) points of inflection. Then graph a quick sketch of the graph.

- **Solution.**
  \[
  \text{Domain, } 10x + 2 \neq 0, \quad x \neq -\frac{1}{5} \quad (-\infty, -\frac{1}{5}), (-\frac{1}{5}, \infty)
  \]

  \[
  k(x) = 0 = x + 2, \quad \text{x-int. } x = -2
  \]

  \[
  k(0) = \frac{0 + 2}{0 + 2} = 1, \quad \text{y-int. } y = 1
  \]

  \[
  \lim_{x \to \infty} \frac{x + 2}{10x + 2} = \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{10 + \frac{2}{x}} = \frac{1}{10} \quad \text{H.A. } y = \frac{1}{10}
  \]

  \[
  \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{10 + \frac{2}{x}} = \frac{1}{10} \quad \text{V.A. } x = -\frac{1}{5} \quad \text{Not on domain. No local max/mins}
  \]

  \[
  k'(x) = \frac{(10x + 2) - 10(x + 2)}{(10x + 2)^2} = \frac{-18}{(10x + 2)^2} \quad \text{critical point at } x = -\frac{1}{5} \quad \text{No POI}
  \]

  \[
  k'(-1) = -\frac{18}{64} = \frac{-360}{360} = -\frac{360}{512}, \quad k'(1) = \frac{-18}{144} = \frac{-360}{512}, \quad f''(-1) = \frac{360}{(-8)^3} = \frac{360}{512}, \quad f''(1) = \frac{360}{(12)^3}
  \]

  \[
  \text{Concave Up} \left( -\frac{1}{5}, \infty \right), \quad \text{Concave Down} \left( -\infty, -\frac{1}{5} \right)
  \]

![Graph](image-url)
Example 20. The manager of a large apartment complex knows from experience that 80 units will be occupied if the rent is 300 dollars per month. A market survey suggests that, on average, one additional unit will remain vacant for each 6 dollar increase in rent. Similarly, one additional unit will be occupied for each 6 dollar decrease in rent. What rent should the manager charge to maximize the revenue?

- **Solution.**
  
  \[ R = q \times p, \quad q = mp + b, \quad m = \frac{q_1 - q_0}{p_1 - p_0} = \frac{79 - 80}{306 - 300} = -\frac{1}{6}, \]
  
  \[ 80 = -\frac{1}{6}(300) + b, \quad b = 130 \]
  
  \[ R = p(-\frac{1}{6}p + 130), \quad R' = 0 = -\frac{1}{3}p + 130, \quad p = \$390 \]
  
  p = $390 to give max revenue

Example 21. Use Newton’s method to approximate a critical point for the function \( \frac{3}{5}x^8 + \frac{6}{5}x^5 + 2x + 7 \) near the point \( x = 2 \).

- **Solution.**
  
  \[ f'(x) = 3x^7 + 6x^4 + 2, \quad f''(x) = 21x^6 + 24x^3 \]
  
  \[ x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}, \quad x_1 = 2, \quad x_2 = 2 - \frac{3(2)^7 + 6(2)^4 + 4}{21(2)^6 + 24(2)^3} = 2 - \frac{482}{1536} = 1.6862 \]
  
  \[ x_3 = 1.6862 - \frac{3(1.6862)^7 + 6(1.6862)^4 + 4}{21(1.6862)^6 + 24(1.6862)^3} = 1.6862 - 0.279 = 1.41 \]
  
  could keep going, but \( x_3 \) is far enough.

Example 22. Consider the function \( f(t) = 4 \sec^2 t - 4t^3 \). Where \( F'(t) = f(t) \) and \( F(0) = 14 \). Determine what \( F(t) \) is.

- **Solution.**
  
  \[ F(t) = 4 \tan(t) - t^4 + C, \quad F(0) = 4 \tan(0) - 0^4 + C = 14, \quad C = 14 \]

  \[ F(t) = 4 \tan(t) - t^4 + 14 \]

Example 23. Give an overestimate and an underestimate of the area between the \( x \)-axis and graph of the function \( f(x) = -5 + 6x - x^2 \) from \( x_1 = 1 \) to \( x_2 = 5 \) using 4 rectangles of equal width.

- **Solution.**
  
  \[ \Delta x = \frac{4}{4} = 1 \]
  
  \( f(1) = -5 + 6 - 1 = 0, \quad f(2) = -5 + 12 - 4 = 3, \quad f(3) = -5 + 18 - 9 = 4, \quad f(4) = -5 + 24 - 16 = 3, \)
  
  \( f(5) = -5 + 30 - 25 = 0 \)

  Underestimate: \( 1(f(1) + f(2) + f(4) + f(5)) = 1(0 + 3 + 4 + 0) = \text{Underestimate: 6} \)

  Overestimate: \( 1(f(2) + f(3) + f(3) + f(4)) = 3 + 4 + 4 + 3 = \text{Overestimate: 14} \)

Example 24. Find the summation of \( \sum_{i=1}^{50} i(6i + 5) \).

- **Solution.**
  
  \[ \sum_{i=1}^{50} 6i^2 + 5i = \frac{50(50+1)(2(50)+1)}{6} + \frac{50(50+1)}{2} = 50(51)(101) + 125(51) = 263925 \]
Example 25. Suppose $\int_6^{15} f(x) \, dx = 2$, $\int_6^{12} f(x) \, dx = 12$, and $\int_{12}^{15} f(x) \, dx = 10$. Find $\int_9^{12} f(x) \, dx$ and $\int_9^9 f(x) \, dx$.

- **Solution.**

\[ \int_9^{12} f(x) \, dx = \int_6^{15} f(x) \, dx - \int_{12}^{15} f(x) \, dx - \int_6^9 f(x) \, dx = 2 - 12 - 10 = -20 \]
\[ \int_9^9 f(x) \, dx = -(\int_6^{15} f(x) \, dx - \int_9^9 f(x) \, dx) = -(2 - 12) = 10 \]

Example 26. Let $f(x) = \int_x^{x^2} 8 \sin^2(t) \, dt$. Find $f'(x)$.

- **Solution.**

$f'(x) = 8 \sin^2(x^2)(2x) - 8 \sin^2(x)$

Example 27. Find the average value of the function $l(x) = 5x^{-2}$ from $[1, 5]$. Then find where the average value occurs.

- **Solution.**

$f_{ave} = \frac{1}{5-1} \int_1^5 5x^{-2} \, dx = \left[ -\frac{1}{4} 5x^{-1} \right]_1^5 = -\frac{1}{4} \left( \frac{5}{5} - \frac{5}{1} \right) = f_{ave} = 1$

$x = 5x^{-2} \quad x^2 = 5 \quad x = \pm \sqrt{5} \quad x = \sqrt{5}$

Example 28. Evaluate the definite integral $\int_{\frac{\pi}{5}}^{\frac{\pi}{2}} \frac{\cos(z)}{\sin^2(z)} \, dz$.

- **Solution.**

$u = \sin(z) \quad du = \cos(z) \, dz$

$\int \frac{1}{u^4} \, du = -4u^{-\frac{1}{4}} \quad -4(\sin(z))^{-\frac{1}{4}} \bigg|_{\frac{\pi}{5}}^{\frac{\pi}{2}} = -4(1 - (\frac{1}{2})^{-\frac{1}{4}}) = -4(1 - 16) = 60$

Example 29. Sketch the region bounded by the curves. Then, decide whether to integrate with respect to $x$ or $y$. Then find the area between the curves. $2y + x = 2$, $y^2 - x = 1$.

- **Solution.**

\[ \int_{-3}^{1} 2 - 2y - (y^2 - 1) \, dy = 2y - y^2 - \frac{1}{3} y^3 + y \bigg|_{-3}^{1} = 2 - 1 - \frac{1}{3} + 1 - (-6 - 9 + 9) = 11 - \frac{1}{3} \]