Name: ______________________________

Section: ____________ **Recitation Instructor**: ____________________

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything excepts pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 12.
- Fill in your name, etc. on this first page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- You will be given exactly 90 minutes for this exam.

I have read and understand the above instructions: ____________________

SIGNATURE

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) Suppose $(1, 2)$ is a critical point of a function f with continuous second derivatives such that:

$$f_{xx}(1, 2) = 1, \quad f_{yy}(1, 2) = 5, \quad f_{xy}(1, 2) = 2$$

   A. f has a local min at $(1, 2)$.
   B. f has a local max at $(1, 2)$.
   C. f has a saddle point at $(1, 2)$.
   D. Not enough information is given.

2. (5 points) The volume of the solid in the first octant bounded by the cylinder $z = 4 - x^2$ and the plane $y = 4$ can be expressed as:

$$\text{I. } \int_0^4 \int_0^4 4 - x^2 \; dy \; dx$$
$$\text{II. } \int_0^2 \int_0^4 \int_0^{4-x^2} 1 \; dz \; dy \; dx$$
$$\text{III. } \int_0^4 \int_0^4 4 - x^2 \; dx \; dy$$

   A. I.
   B. II.
   C. III.
   D. I. and III.
   E. None of the above

3. (5 points) A parametric representation for the part of the plane $z = 8$ that lies inside the cylinder $x^2 + y^2 = 9$ can be given by:

   A. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 8\rangle$, $\quad s \in [0,3], t \in [0,2\pi]$.

   B. $\mathbf{r}(s,t) = \langle t\cos s, t\sin s, 8\rangle$, $\quad s \in [0,2\pi], t \in [0,3]$.

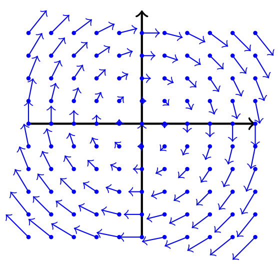
   C. $\mathbf{r}(s,t) = \langle -s\sin t, s\cos t, 8\rangle$, $\quad s \in [0,3], t \in [0,2\pi]$.

   D. All of the above.

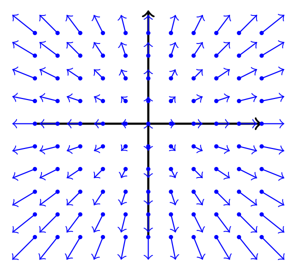
   E. None of the above.

4. (5 points) Which of the following plots could be $\mathbf{F} = \langle y, x - y\rangle$.

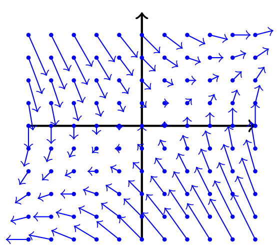
   A.



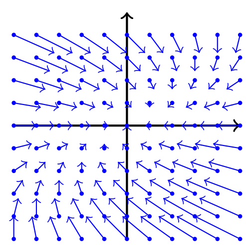
   C.



   B. ← Answer



   D.



   E. None of the above.

Extra Work Space.

Fill in the Blanks. No work needed. Possible scores are 0,3,5 based soley on answer.

5. (10 points) The surface area of the part of the surface $z = xy^2$ that lies above the triangle in the $xy-$plane with verticies $(0,0), (1,0)$, and $(0,2)$ can be expressed by $\int_{x_1}^{x_2} \int_{y_1}^{y_2} g(x,y)\ dy\ dx$ where:

$g(x,y) =$ $\sqrt{1 + (y^2)^2 + (2xy)^2}$

$x_1 =$ 0

$x_2 =$ 1

$y_1 =$ 0

$y_2 =$ $2 - 2x$

Extra Work Space.

6. Suppose $\mathbf{F} = \langle 10e^x \sin y, 5e^x \cos y, 4z \rangle$.

   (a) (5 points) curl $\mathbf{F}$ = $\langle 0, 0, -5e^3 \cos y \rangle$.

   (b) (5 points) div $\mathbf{F}$ = $5e^3 \sin y + 4$.

7. (5 points) Setup a line integral for $\int_C xyz\ ds$ where C is parametrized by $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 3 \rangle, 0 \le t \le \pi$. Simplify as much as possible but **do not evaluate the integral**.

$$\int_C f\ ds = \int_0^\pi (\cos(2t))(\sin(2t))(3)\sqrt{4}\ dt$$

8. (5 points) If a smooth surface S is given parametrically as $\mathbf{r}(s,t)$ with (s,t) in D then its surface area is given by the formula:

$$A(S) = \iint_D |\mathbf{r}_s \times \mathbf{r}_t|\ ds\ dt$$

Extra Work Space.

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

9. Suppose $f(x,y) = \dfrac{y^2}{x}$.

   (a) (6 points) Find the rate of change of f at $P(1,2)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$.

Solution: First let's collect the components to our answer

$$\begin{aligned} \nabla f &= \left\langle \frac{-y^2}{x^2}, \frac{2y}{x} \right\rangle \\ \nabla f(P) &= \langle -4, 4 \rangle \\ \mathbf{u} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 2,3 \rangle}{\sqrt{13}} \end{aligned}$$

Now we assemble correctly

$$\begin{aligned} D_{\mathbf{v}} f(P) &= \langle -4, 4 \rangle \cdot \frac{\langle 2,3 \rangle}{\sqrt{13}} \\ &= \frac{-8+12}{\sqrt{13}} = \boxed{\frac{4}{\sqrt{13}}} \end{aligned}$$

   (b) (6 points) Find the minimum rate of change of f at $P(1,2)$.

Solution: The minimum rate of change is given by $-|\nabla f(P)|$. Using some work from (a) we see that the answer should be:

$$\begin{aligned} \nabla f(P) &= \langle -4, 4 \rangle \\ |\nabla f(P)| &= 4\sqrt{2} \\ -|\nabla f(P)| &= \boxed{-4\sqrt{2}} \end{aligned}$$

   (c) (6 points) Which direction does the minimum rate of change occur?

Solution: The minimum rate of change occurs in the direction $-\nabla f(P)$. Which is:

$$\begin{aligned} \nabla f(P) &= \langle -4, 4 \rangle \\ -\nabla f(P) &= \boxed{\langle 4, -4 \rangle} \end{aligned}$$

10. (14 points) Suppose that $f(x, y) = x^2 + y^2$ and D is the region bounded by $y = x + 1$ and $y = x^2 - 1$. Find the absolute minimum and maximum of f on D. Show necessary work.

Solution: To show this we must collect all the critical points (on the interior of D and on the boundary) along with all the endpoints of the boundary.

Crit points on interior
$f_x = 2x$ and $f_y = 2y$ giving us the only crit point at $(0, 0)$.

Crit points on $y = x + 1$
$f(x, x+1) = x^2 + (x+1)^2 = 2x^2 + 2x + 1$. Giving us the derivative $f'(x) = 4x + 2$. So there is a crit point at $(-1/2, 1/2)$.

Crit points on $y = x^2 - 1$
$f(x, x^2-1) = x^2 + (x^2-1)^2 = x^4 - x^2 + 1$. Giving us the derivative $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$. So there are a crit points at $(0, -1), (1/\sqrt{2}, -1/2)$, and $(-1/\sqrt{2}, -1/2)$.

End points of boundary
Solving $x + 1 = x^2 - 1$ we get $0 = x^2 - x - 2 = (x+1)(x-2)$ giving us the endpoints $(-1, 0)$ and $(2, 3)$.

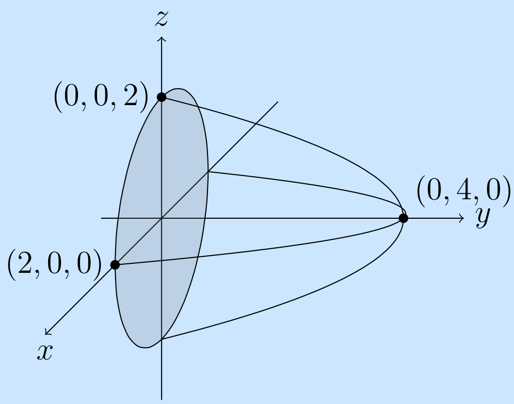
Final evaluation
Plugging in all these points of interest to f we can identify the absolute maximum and minimum.

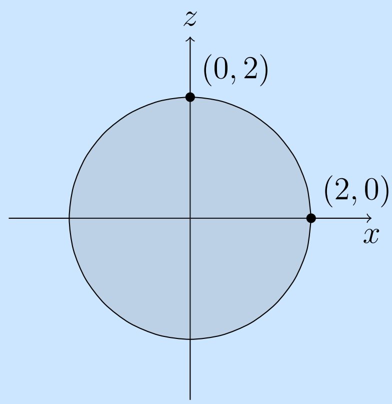
$$\begin{aligned}
f(0,0) &= 0 + 0 = 0 \longleftarrow \text{Absolute Min} \\
f(-1/2, 1/2) &= 1/4 + 1/4 = 1/2 \\
f(0,-1) &= 0 + 1 = 1 \\
f(1/\sqrt{2}, -1/2) &= 1/2 + 1/4 = 3/4 \\
f(-1/\sqrt{2}, -1/2) &= 1/2 + 1/4 = 3/4 \\
f(-1, 0) &= 1 + 0 = 1 \\
f(2, 3) &= 4 + 9 = 13 \longleftarrow \text{Absolute Max}
\end{aligned}$$

11. Express $\iiint_E f(x,y,z)\,dV$ as an iterated integral in the two different ways below, where E is the solid bounded by the surfaces $y = 4 - x^2 - z^2$ and $y = 0$. **(Find the limits of integration).**

(a) (8 points) $\iiint f(x,y,z)\,dy\,dz\,dx$

Solution: Sketching some traces we get the 3D picture below. Also since we will integrate with respect y first in this case lets "smash" into the xz-plane to get the corresponding 2D picture.



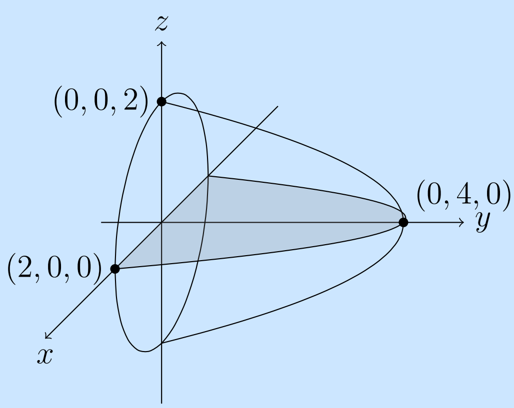


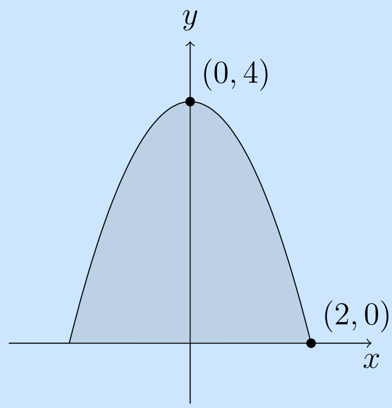
This should be enough to give us the final answer of:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-z^2} f(x,y,z)\,dy\,dz\,dx$$

(b) (8 points) $\iiint f(x,y,z)\,dz\,dy\,dx$

Solution: Sketching some traces we get the 3D picture below. Also since we will integrate with respect z first in this case lets "smash" into the xy-plane to get the corresponding 2D picture.



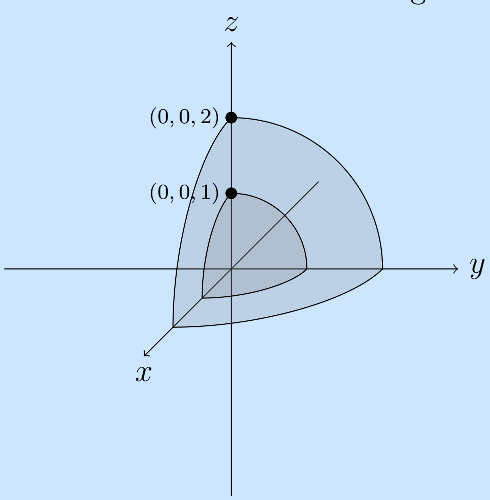


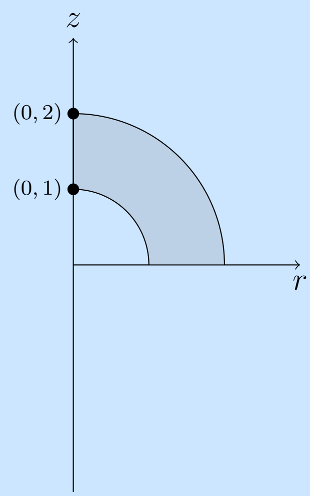
This should be enough to give us the final answer of:

$$\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} f(x,y,z)\,dz\,dy\,dx$$

12. (14 points) Evaluate the triple integral $\iiint_E z\ dV$ where E is the region in the first octant that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Solution: Let's sketch the region of integration in 3D and in the rz-halfplane :





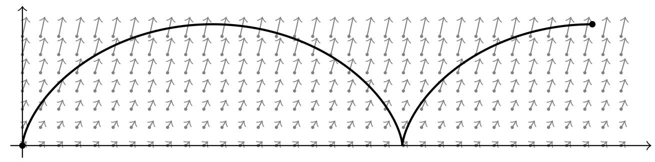
Hopefully all these spheres around clearly suggest spherical coordinates is the way to go. Bounding between these spheres gives us $1 \leq \rho \leq 2$. Being in the first octant gives us $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$. Therefore our integral is as follows:

$$\begin{aligned}
\iiint_E z\ dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho\cos\phi)(\rho^2 \sin\phi)\ d\rho\ d\phi\ d\theta \\
&= \frac{\pi}{2} \int_0^{\pi/2} \int_1^2 \rho^3 \sin\phi\cos\phi\ d\rho\ d\phi \\
&= \frac{\pi}{2} \int_0^{\pi/2} \left[\frac{\rho^4}{4} \sin\phi\cos\phi\right]_1^2\ d\phi \\
&= \frac{\pi}{2} \int_0^{\pi/2} \left[\frac{15}{4} \sin\phi\cos\phi\right]\ d\phi \\
&= \frac{\pi}{2} \left[\frac{15}{4} \frac{\sin^2\phi}{2}\right]_0^{\pi/2} \\
&= \frac{\pi}{2} \left[\frac{15}{4} \frac{1}{2}\right] = \boxed{\frac{15\pi}{16}}
\end{aligned}$$

13. (10 points) Find the work done by the force field $\mathbf{F}(x,y) = \mathbf{i} + (2y+1)\mathbf{j}$ in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \le t \le 3\pi$$

depicted to the right.



Solution:

$$\begin{aligned}
\text{Work } &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{3\pi} \langle 1, 2y+1 \rangle \cdot \langle 1 - \cos t, \sin t \rangle \; dt \\
&= \int_0^{3\pi} [1 - \cos t + (2(1-\cos t) + 1)\sin t] \; dt \\
&= \int_0^{3\pi} [1 - \cos t + 3\sin t - 2\cos t \sin t] \; dt \\
&= [t - \sin t - 3\cos t - \sin^2 t]_0^{3\pi} \\
&= [3\pi - 3(-1) + 3] = \boxed{6 + 3\pi}
\end{aligned}$$

Alternatively one can notice that the vector field is conservative and calculate the integral in this way.

14. (10 points) Use Greens Theorem to evaluate $\oint_C x^2y \; dx - xy^2 \; dy$, where C is the circle $x^2 + y^2 = 4$.

Solution: The vector field is $\mathbf{F} = \langle x^2y, -xy^2 \rangle$. So by Green's theorem we have:

$$\begin{aligned}
\oint_C x^2y \; dx - xy^2 \; dy &= \iint_D -y^2 - x^2 \; dy \; dx \\
&= -\int_0^{2\pi} \int_0^2 r^2 r \; dr \; d\theta \\
&= -2\pi \left[\frac{r^4}{4}\right]_0^2 = \boxed{-8\pi}
\end{aligned}$$

15. Consider the vector field $\mathbf{F}(x,y) = (7ye^{7x})\mathbf{i} + (e^{7x} + 2y)\mathbf{j}$

(a) (10 points) Use a systematic approach to find a potential function for $\mathbf{F}$. Even if you can do it in your head, instead show work.

Solution:

$$\begin{aligned} f_x &= 7ye^{7x} \\ f &= ye^{7x} + g(y) \\ f_y &= e^{7x} + g'(y) \\ e^{7x} + 2y &= e^{7x} + g'(y) \\ 2y &= g'(y) \\ y^2 + K &= g(y) \\ \implies f &= \boxed{ye^{7x} + y^2 + K} \end{aligned}$$

(b) (10 points) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is parametrized by $\mathbf{r}(t) = \cos t\,\mathbf{i} + t\mathbf{j}, \quad t \in [0, 2\pi]$.

Solution: $A = \mathbf{r}(0) = \langle 1, 0 \rangle$ and $B = \mathbf{r}(2\pi) = \langle 1, 2\pi \rangle$ so by the Fundamental Theorem of Line Integrals we have:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(B) - f(A) \\ &= f(1, 2\pi) - f(1, 0) \\ &= \boxed{2\pi e^7 + 4\pi^2} \end{aligned}$$

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	10	
3	10	
4	10	
5	20	
6	18	
7	14	
8	16	
9	14	
10	20	
11	20	
Total:	152	