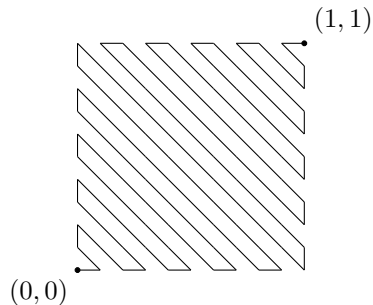


3 The Fundamental Theorem for Line Integrals

Warm up:

Calculate the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ from $(0, 0)$ to $(1, 1)$ in the vector field $\mathbf{F} = \langle 2x, 2y \rangle$.

1. Where C is parametrized by $\langle t, t \rangle \quad t \in [0, 1]$
2. Where C is parametrized by $\langle t, t^2 \rangle \quad t \in [0, 1]$
3. Where C is parametrized by $\langle \sin\left(\frac{\pi t}{2}\right), t^2 \rangle \quad t \in [0, 1]$
4. Where C is parametrized by the picture:



After doing 1,2,3 hopefully we would all conjecture the answer to 4.

So what is so special about the vector field $\mathbf{F} = \langle 2x, 2y \rangle$? It seems like no matter the path we choose we always get the same answer! (How nice). This has a nice name:

Definition(s) 3.1.

Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all curves. Then the integral is called _____ in D .

And so we need to develop some mathematics to help us know when a field is going to be path independent or not. It turns out that _____ fields are _____.

Theorem 3.2.

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then $\mathbf{F}$ is a conservative vector field on D ; that is, there exists a function ____ such that _____.

Definition(s) 3.3.

1. A region D is **open** if for every point P in D there is a disk with center P that lies entirely in D . (So D doesn't contain any of its boundary points.)
2. A region D is **connected** if any two points in D can be joined by a path that lies in D .
3. A curve is called **closed** if its terminal point is the same as its initial point.
4. A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints.
5. A **simply-connected region** in a plane is a connected region D such that every simple closed curve in D encloses only points that are in D .

Looking at $\mathbf{F} = \langle 2x, 2y \rangle$ again. Can you find any functions $f(x, y)$ who have gradient $\langle 2x, 2y \rangle$?

Moreover please notice that

Thats to say that in our case

Theorem 3.4 (Fundamental Theorem of Line Integrals).

Let C be a smooth curve joining the point A to the point B in the plane or space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on the domain D containing C . Then:

equivalently:

Theorem 3.5.

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = _$ for every closed path C in D .

So here is where we sit:

1. We like conservative vector fields because they are path independent.
2. We like them even more because if we can find their potential function then line integrals are extremely easy to calculate.

Here are the natural questions we need to ask:

1. Given a vector field is there a way to tell if it is _____?
2. Okay we know we have a conservative vector field... How can we find the _____?

and very importantly we have:

Theorem 3.6 (Component Test).

Let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order derivatives and

Then $\mathbf{F}$ is _____.

Finally! it's example!!!!

Example 3.7.

Consider the vector field $\mathbf{F} = \langle e^x \cos y + y, x - e^x \sin y + 3 \rangle$.

1. Show that $\mathbf{F}$ is conservative over its natural domain
2. Find a potential function for $\mathbf{F}$.
3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parametrized by $\mathbf{r}(t) = \langle t, \sin t \rangle$ and $t \in [0, 1]$

Remark 3.8 (Technique for finding potential function).

1. Integrate _____ to get _____.
2. Try to solve for _____
 - (a) Differentiate _____ with respect to ____ and set it equal to Q .
 - (b) Solve for _____.
 - (c) Integrate _____ with respect to ____ to get $g(y)$.

Example 3.9.

1. Show that for $\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$ we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

2. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ where C is a loop parametrized by $r(t) = \langle \cos t, \sin t \rangle$, $t \in [0, 2\pi]$

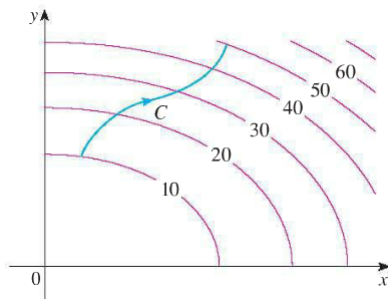
Solution.

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle \cdot \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} \left\langle \frac{-\sin t}{1}, \frac{\cos t}{1} \right\rangle \cdot \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} 1 dt = \boxed{2\pi}
 \end{aligned}$$

3. Does this contradict **Theorem 3.5**?

Group Work

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. Consider the vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$

(a) Determine that $\mathbf{F}$ is conservative using the component test.

(b) Find a function f such that $\mathbf{F} = \nabla f$

(c) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \leq t \leq \pi$.

Note: We currently do not have the correct theory in place to show that a vector field of 3 variables is conservative. However if we assume it is conservative then we can find potential functions. Here is an example from the book (on page 1104) of how to do this for 3 variables. Please read this through if you have trouble with your WeBWorK homework assignment.



EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f , then

$$\boxed{11} \quad f_x(x, y, z) = y^2$$

$$\boxed{12} \quad f_y(x, y, z) = 2xy + e^{3z}$$

$$\boxed{13} \quad f_z(x, y, z) = 3ye^{3z}$$

Integrating $\boxed{11}$ with respect to x , we get

$$\boxed{14} \quad f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x . Then differentiating $\boxed{14}$ with respect to y , we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with $\boxed{12}$ gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite $\boxed{14}$ as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with $\boxed{13}$, we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■