1 Vector Fields

Definition(s) 1.1.

1. Let D be a set in $\mathbb{R}^2$ (plane region). A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x,y)$.
2. Let E be a set in $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x,y,z)$.

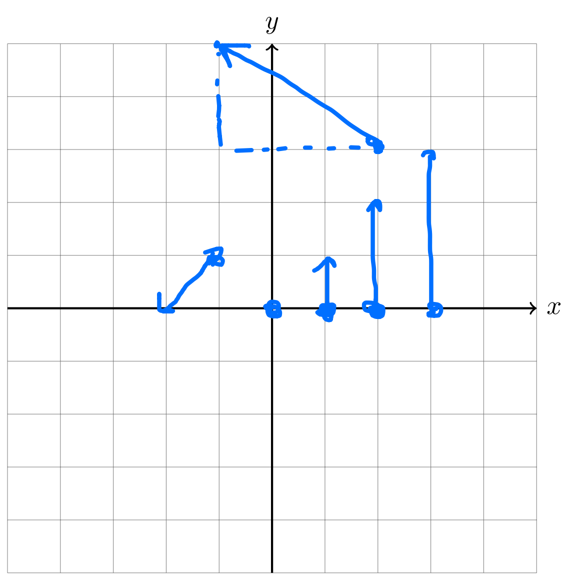
Let's practice by sketching a vector field.

Example 1.2.

Sketch the vector field:

$$\mathbf{F}(x, y) = \langle -y, x \rangle$$

on the graph below.



$\vec{F}(0,0) = \langle -0, 0 \rangle$

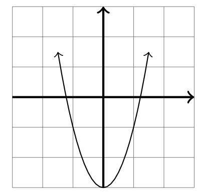
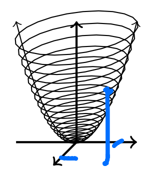
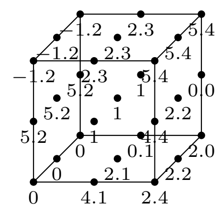
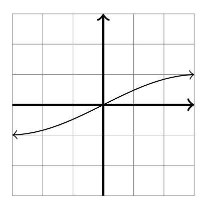
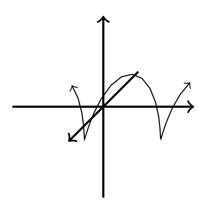
$F(1,0) = \langle 0, 1 \rangle$

$F(2,0) = \langle 0, 2 \rangle$

$F(2,3) = \langle -3, 2 \rangle$

$F(-1,1) = \langle -1, -1 \rangle$

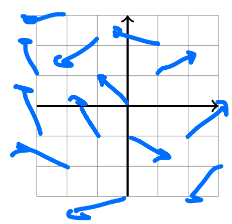
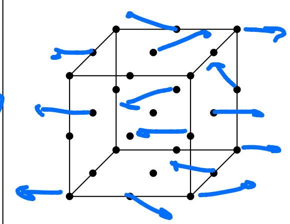
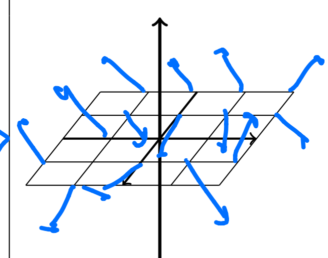
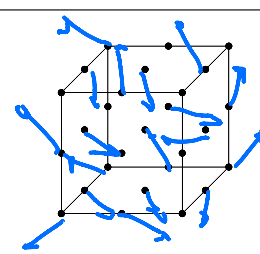
Today we are upgrading to a new level of function:
Let's talk about dimensions really quick:
So far we have function that take:

Dimensions	Equation	Example	Picture
1 dimension $\to$ 1 dimension	$y=f(x)$ or $x=x(t)$	$y=2x^2-3$ or $x(t)=3t-1$	
2 dimensions $\to$ 1 dimension	$z=f(x,y)$	$z=x^2+y^2$	
3 dimensions $\to$ 1 dimension	$w=f(x,y,z)$	$T=xy^2+2z$	
1 dimension $\to$ 2 dimensions	$\vec{r}(t)=\langle x(t),y(t)\rangle$	$\vec{r}(t)=\langle 2t,\sin t\rangle$	
1 dimension $\to$ 3 dimensions	$\vec{r}(t) = \langle x(t),y(t),z(t)\rangle$	$\vec{r}(t) = \langle \frac{2t}{5},\sin t,\cos t\rangle$	

		to Dim.		
		1	2	3
from Dim.	1	X	X	X
	2	X	O	O
	3	X	O	O

and so you can see we are missing some pieces still!

These missing pieces will be called **vector fields** and they are our friends for the rest of the semester.

Dimensions	Equation	Example	Picture
2 dimension $\rightarrow$ 2 dimension	$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$	$\vec{F}(x,y) = \langle x^2y, -\frac{y}{x} \rangle$	
Rare 3 dimension $\rightarrow$ 2 dimension	$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z) \rangle$	$\vec{F}(x,y,z) = \langle xyz^2, \frac{\tan z}{x} \rangle$	
Rare 2 dimension $\rightarrow$ 3 dimension	$\vec{F}(x,y) = \langle P(x,y), Q(x,y), R(x,y) \rangle$	$\vec{F}(x,y) = \langle xy, x^4, \tan y \rangle$	
3 dimension $\rightarrow$ 3 dimension	$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$	$\vec{F}(x,y,z) = \langle x\cos y, yz, -xz \rangle$	

Here are some pretty pictures from the book.

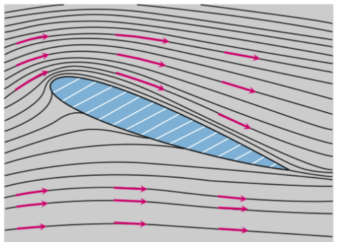


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

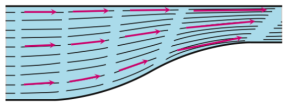


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

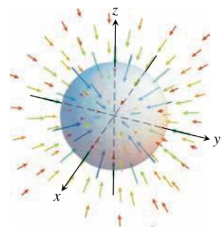
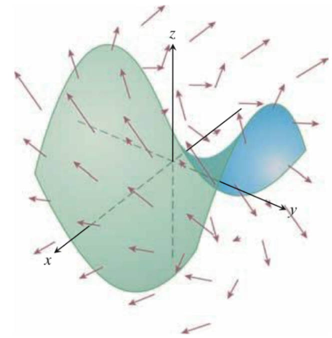


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.



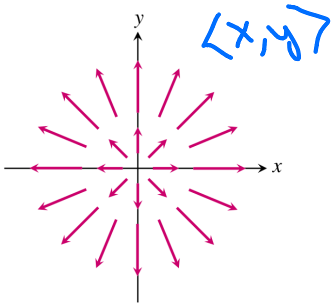


FIGURE 16.11 The radial field

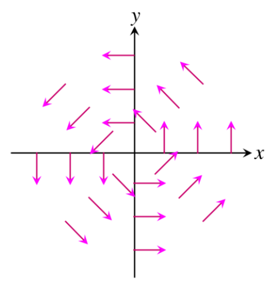


FIGURE 16.12 A “spin” field of

Vector fields can represent many different things. The main applications we will focus on are:

1. Force
2. Velocity

We have technically seen vector fields before even though we never used it's full potential. Any guesses?

Gradient

Definition(s) 1.3.

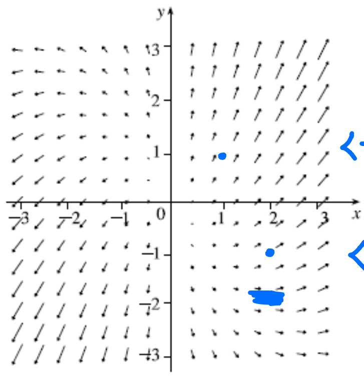
1. A gradient vector field is vector field found by taking the gradient of a function.

   Ex: $f(x,y) = x^2 + y^2 \Rightarrow \nabla f = \langle 2x, 2y \rangle$

2. A vector field **F** is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $\nabla f = \vec{F}$. In this situation f is called a potential function for **F**.

   Ex: $\vec{F} = \langle x, y \rangle$ is conservative with potential function $f = \frac{x^2}{2} + \frac{y^2}{2}$

Example 1.4.



$\langle +, + \rangle$

$\langle +, + \rangle$

Figure may be scaled down

Which of the vector field describes the plot to the left?

	at $(1,1)$	at $(2,-1)$
1. $\cancel{\langle x, x - y\rangle}$	$\langle 1, 0\rangle$	
2. $\cancel{\langle y, x - y\rangle}$	$\langle 1, 0\rangle$	
3. $\boxed{\langle x, x + y\rangle}$	$\langle 1, 2\rangle$	$\langle 2, 1\rangle$
4. $\cancel{\langle y, x + y\rangle}$	$\langle 1, 2\rangle$	$\langle -1, 1\rangle$

Example 1.5.

Find the gradient vector field of $f(x, y) = 2xy + 3x - e^{-xy}$

$$\nabla f = \left\langle 2y + 3 - e^{-xy}(-y),\ 2x + 0 - e^{-xy}(-x) \right\rangle$$

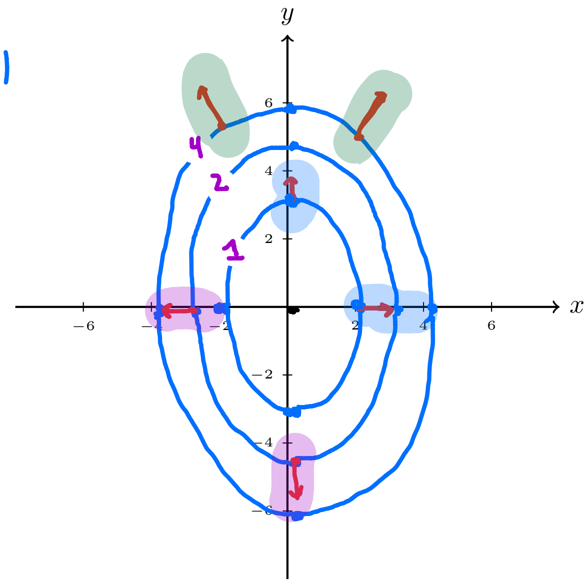
$$= \left\langle 2y + 3 + ye^{-xy},\ 2x + xe^{-xy} \right\rangle$$

Group Work

1. For each of the following functions, draw level curves $f(x, y) = k$ for the indicated values of k. Then compute the gradient vector field, and sketch it at one or two points on each level curve.

   (a) $f(x, y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}; k = 1, 2, 4$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$



$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle$

$\nabla f(2,0) = \langle 1, 0 \rangle$

$\nabla f(0,3) = \langle 0, 2/3 \rangle$

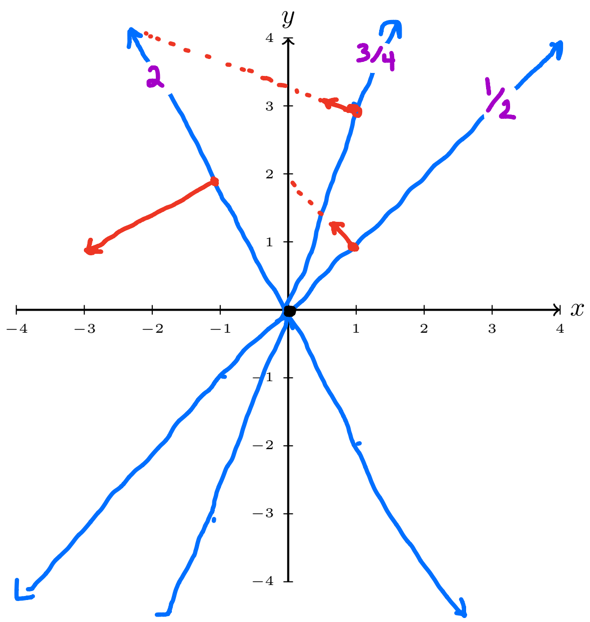
$\nabla f(-\sqrt{8},0) = \langle -\sqrt{2}, 0 \rangle$

$\nabla f(0,-\sqrt{18}) = \langle 0, -\frac{2}{3}\sqrt{2} \rangle$

$\nabla f(2,\sqrt{27}) = \langle 1, \frac{2}{3}\sqrt{3} \rangle$

$\nabla f(-2,\sqrt{27}) = \langle -1, \frac{2}{3}\sqrt{3} \rangle$

   (b) $f(x, y) = \dfrac{y}{x + y}, x \neq -y; k = 1/2, 3/4, 2$



$\frac{1}{2} = \frac{y}{x+y} \Rightarrow 2y = x + y \Rightarrow y = x$

$\frac{3}{4} = \frac{y}{x+y} \Rightarrow 4y = 3x + 3y \Rightarrow y = 3x$

$2 = \frac{y}{x+y} \Rightarrow y = 2x + 2y \Rightarrow y = -2x$

$\nabla f = \left\langle \frac{-y}{(x+y)^2}, \frac{x}{(x+y)^2} \right\rangle$

$\nabla f(1,1) = \left\langle \frac{-1}{4}, \frac{1}{4} \right\rangle$

$\nabla f(1,3) = \left\langle \frac{-3}{16}, \frac{1}{16} \right\rangle$

$\nabla f(-1,2) = \left\langle \frac{-2}{1}, \frac{-1}{1} \right\rangle$

2 Line Integrals

Remark 2.1 (Things to remember from past sections)**.**

1. A vector representation of a line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by
$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \le t \le 1$$
$r_0 + (r_1 - r_0)t$
2. Arc length function $s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$
3. Taking the derivative with respect to t we get
$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
4. We can express this as $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Theorem 2.2.
The arc length of a curve C parametrized by $\mathbf{r}(t)$ is given by:

$$\int_C 1\, ds$$

This is a natural ideal because now all our measure of "volume" can be written as integrals of 1.

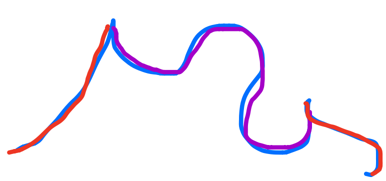
1. Volume of $E =$ $\iiint_E 1\, dV$
2. Area of $R =$ $\iint_R 1\, dA$
3. Length of $C =$ $\int_C 1\, ds$

And just as in CH15 we integrated other functions in our double in triple integrals we also would like to integrate other functions here in our line integrals.

Definition(s) 2.3.
If f is defined on a smooth curve C, then the line integral **of** f **along** C is given by:

$$\int_C f(x,y)\, ds = \int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: smooth curves are defined in 13.3 as continuous and having a parametrization $\mathbf{r}(t)$ such that $\mathbf{r}'(t) \neq 0$.



Although we are defining these as planar curves we can easily upgrade to space curves by "sprinkling in zs"

Definition(s) 2.4.

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt$$

Notice in both of these cases

$$\int_C f\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\ dt$$

Example 2.5.

Suppose you are a whale who is eating plankton as he/she swims through the ocean. The plankton are spread all throughout the ocean with a function $p(x,y,z) = -\frac{1}{\pi}(10+z+x)$. You (the whale) are chilling out at $(1,0,-12)$ are about to swim around in a circular curve; $C:\ x^2+y^2=1,\ z=-12$
How many plankton do you eat?

$$\int_C p(x,y,z)\,ds$$

$$r(t) = \langle \cos t, \sin t, -12\rangle, \quad t\in[0,2\pi]$$

$$|r'(t)| = |\langle -\sin t, \cos t, 0\rangle|$$

$$= \sqrt{\sin^2 t+\cos^2 t+0} = 1$$

$$\int_C p(x,y,z)ds = \int_0^{2\pi} -\frac{1}{\pi}(10-12+\cos t)\,1\,dt$$

$$\sin t\Big|_0^{2\pi} = 0$$

$$= -\frac{1}{\pi}\int_0^{2\pi} \cancel{\cos t} - 2\,dt = -\frac{1}{\pi}\Big[-2t\Big]_0^{2\pi}$$

$$= -\frac{1}{\pi}\Big[-2(2\pi)\Big]$$

$$= \boxed{4}$$

Now let's learn two other cool things line integrals are good for.

How many bricks it takes to build the Great wall of China - Or area under space curves

So the book has a beautiful picture of this:

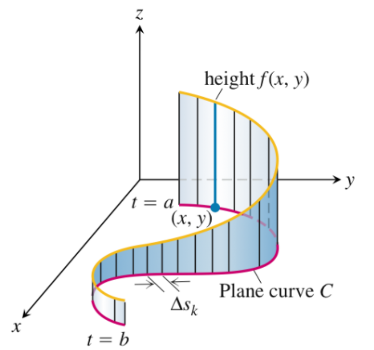


FIGURE 16.5 The line integral $\int_C f\, ds$ gives the area of the portion of the cylindrical surface or "wall" beneath $z = f(x, y) \geq 0$.

Example 2.6.

A portion of the great wall of china can be parametrized by $r(t) = \langle 2\sin t, 2\cos t\rangle$ $t \in [0, \pi]$ where the height is given by $H(x, y) = xy^2$ meters. Each brick has a cross-sectional area of $100\ cm^2$. How many bricks are needed to build this portion of the great wall of china?

$$\int_C H\, ds$$

$$r(t) = \langle 2\sin t, 2\cos t\rangle \quad t \in [0, \pi]$$

$$|r'(t)| = |\langle 2\cos t, -2\sin t\rangle|$$

$$= \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$

$$\int_C H\, ds = \int_0^\pi (2\sin t)(4\cos^2 t)(2\, dt)$$

$$= 16\int_0^\pi \sin t \cos^2 t\, dt$$

$$= -\frac{16}{3}\left[\cos^3 t\right]_0^\pi$$

$$= \frac{16}{3} + \frac{16}{3} = \frac{32}{3}\cancel{m^2} \cdot \frac{100\cancel{cm}}{\cancel{m}} \cdot \frac{\cancel{100cm}}{\cancel{m}} \cdot \frac{1\text{brick}}{\cancel{100cm^2}} = \frac{3200}{3} = 1067 \text{ bricks}$$

How heavy stuff is (given a density function)

Example 2.7.

Suppose I have a nice spring that seems to follow the curve $r(t) = \langle 3\sin t, 3\cos t, 4t\rangle$ with $t \in [0, 8\pi]$ which happens to have a density function of $\delta(x, y, z) = 10 - x - y\ g/cm$. How heavy is the spring?

or ρ if you prefer.

$$\int_C \delta\, ds = \int_0^{8\pi} (10 - 3\sin t - 3\cos t)\sqrt{9\sin^2 t + 9\cos^2 t + 16}\, dt$$

$$= 5\int_0^{8\pi} (10 - 3\sin t - 3\cos t)\, dt$$

$$= 5\left[10t\right]_0^{8\pi} = 400\pi \text{ grams}$$

Now let's consider a special case where $f(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. This gives us a way to do line integrals over vector fields!

Definition(s) 2.8.
Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along** C is

$$\int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\, dt = \int_C \vec{F}\cdot\vec{T}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot r'(t)\, dt \longleftarrow \text{working def}$$

$$= \int_C \vec{F}\cdot d\vec{r}$$

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ then

$$= \int_C P\,dx + Q\,dy$$

If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then

$$= \int_C P\,dx + Q\,dy + R\,dz$$

So besides a great way to torture math students what is this used for? Work

Theorem 2.9.
Take $\mathbf{F}$ to be a force field then the work done by the field over the curve C is given by

$$W = \int_C \vec{F}\cdot\vec{T}\, ds$$

Idea of proof:
Recall

$$W = \vec{F}\cdot\vec{D} \qquad \text{(where } \mathbf{D} \text{ is displacement vector)}$$

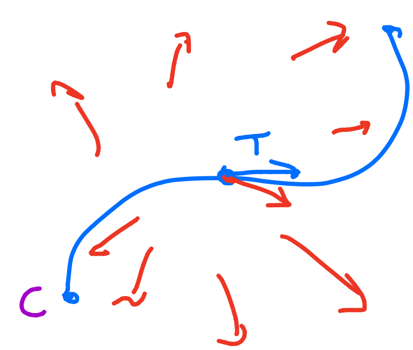
So if we consider the work that the force field is doing at each point on the curve we have:

$\measuredangle < 90 \Rightarrow +$
$\measuredangle > 90 \Rightarrow -$

$$W_{\text{at a point}} = \vec{F}\cdot\vec{T} = |F||T|\cos\theta \qquad \text{(see picture for idea)}$$

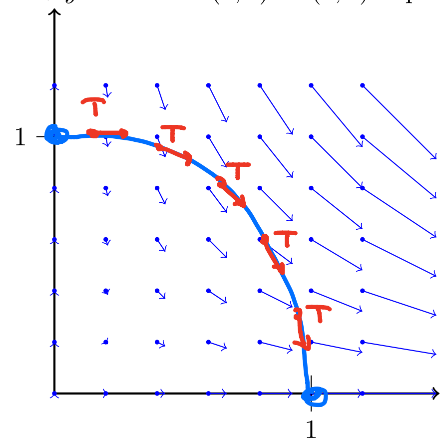
Summing these all up over the curve we get

$$W_{\text{over the curve}} = \int_C \vec{F}\cdot\vec{T}\, ds$$



Example 2.10.

A picture of the force field $\mathbf{F}(x, y)$ is given below. Determine if the work in moving a particle along the quarter circle $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$ is positive or negative using the picture.



Helping

so work is positive

$$\vec{F} \cdot \vec{T} = |F||T| \cos\theta \quad \begin{cases} \measuredangle < 90 \Rightarrow + \\ \measuredangle > 90 \Rightarrow - \end{cases}$$

Example 2.10 (again).

Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, -xy \rangle$ in moving a particle along the quarter circle $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$.

$$\text{Work} = \int_C F \cdot T \, ds = \int_a^b F(r(t)) \cdot r'(t) \, dt$$

$$r(t) = \langle \cos t, \sin t \rangle \qquad t \in \left[\tfrac{\pi}{2}, 0\right]$$

$$r'(t) = \langle -\sin t, \cos t \rangle$$

$$= \int_{\pi/2}^{0} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{\pi/2}^{0} (-\sin t \cos^2 t - \cos^2 t \sin t) \, dt$$

$$= -2 \int_{\pi/2}^{0} \sin t \cos^2 t \, dt$$

$$= \frac{2}{3}\left[\cos^3 t\right]_{\pi/2}^{0} = \frac{2}{3}[1 - 0] = \frac{2}{3}$$

Group Work

1. Evaluate $\int_C y\ dx + z\ dy + x\ dz$ where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

$C_1: \quad r_1(t) = \langle 2+t, 4t, 5t \rangle, \quad t \in [0,1] \implies dr_1 = \langle 1,4,5 \rangle$

$C_2: \quad r_2(t) = \langle 3, 4, 5-5t \rangle, \quad t \in [0,1] \implies dr_2 = \langle 0,0,-5 \rangle$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 \left[(4t)(1) + (5t)(4) + (2+t)(5)\right] dt$$

$$+ \int_0^1 \left[(4)(0) + (5-5t)(0) + (3)(-5)\right] dt$$

$$= \int_0^1 \left[4t + 20t + 10 + 5t - 15\right] dt$$

$$= \int_0^1 \left[29t - 5\right] dt = \left[\frac{29}{2}t^2 - 5t\right]_0^1$$

$$= \frac{29}{2} - 5 = \boxed{19/2}$$

2. What is the calculation in 1. telling you (in terms of Work)?

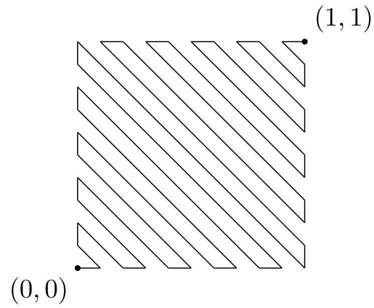
The vector field is helping the particle move across the curve.

3 The Fundamental Theorem for Line Integrals

Warm up:

Calculate the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ from $(0,0)$ to $(1,1)$ in the vector field $\mathbf{F} = \langle 2x, 2y \rangle$.

1. Where C is parametrized by $\langle t, t \rangle \quad t \in [0,1]$
2. Where C is parametrized by $\langle t, t^2 \rangle \quad t \in [0,1]$
3. Where C is parametrized by $\langle \sin\left(\frac{\pi t}{2}\right), t^2 \rangle \quad t \in [0,1]$
4. Where C is parametrized by the picture:



① $\int_a^b F(r(t)) \cdot r'(t)\, dt \qquad r'(t) = \langle 1, 1 \rangle$

$$\int_0^1 \langle 2t, 2t \rangle \cdot \langle 1, 1 \rangle\, dt = \int_0^1 4t\, dt = \left[2t^2\right]_0^1 = 2$$

② ——

③ $r'(t) = \langle \cos\left(\frac{\pi t}{2}\right)\left(\frac{\pi}{2}\right), 2t \rangle$

$$\int_0^1 \langle 2\sin\left(\tfrac{\pi t}{2}\right), 2t^2 \rangle \cdot \langle \tfrac{\pi}{2}\cos\left(\tfrac{\pi t}{2}\right), 2t \rangle\, dt$$

$$\int_0^1 \pi \sin\left(\tfrac{\pi t}{2}\right)\cos\left(\tfrac{\pi t}{2}\right) + 4t^3\, dt$$

$$\left[\sin^2\left(\tfrac{\pi t}{2}\right) + t^4\right]_0^1 = 1 + 1 - 0 - 0 = 2$$

After doing 1,2,3 hopefully we would all conjecture the answer to 4.

So what is so special about the vector field $\mathbf{F} = \langle 2x, 2y \rangle$? It seems like no matter the path we choose we always get the same answer! (How nice). This has a nice name:

Definition(s) 3.1.

Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all curves. Then the integral is called path independent in D.

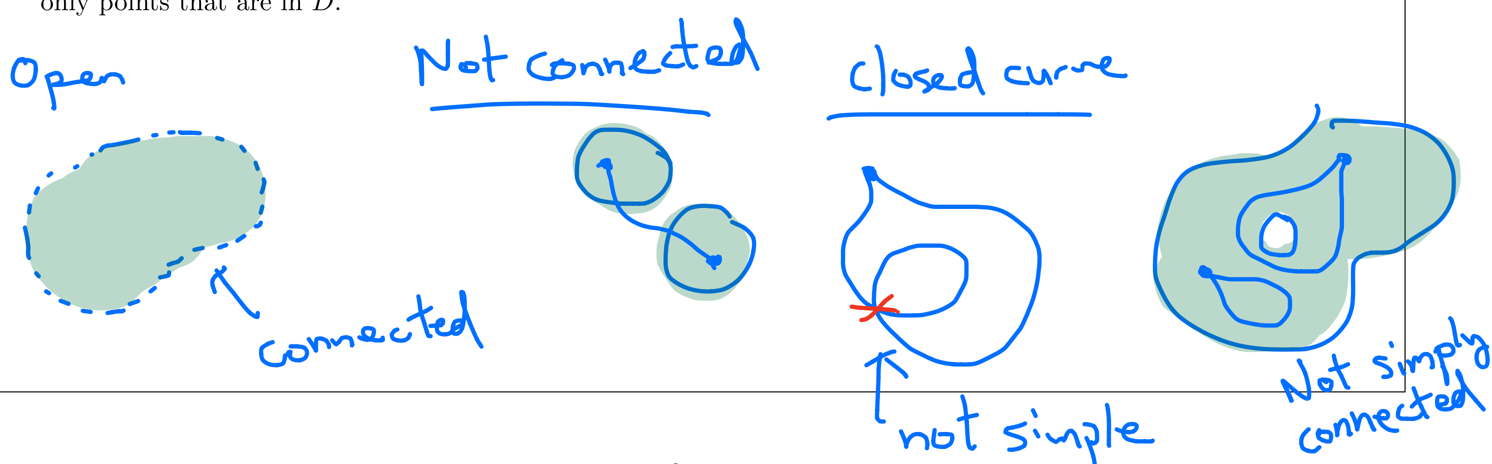
And so we need to develop some mathematics to help us know when a field is going to be path independent or not. It turns out that path independent fields are Conservative.

Theorem 3.2.

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

Definition(s) 3.3.

1. A region D is **open** if for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.)
2. A region D is **connected** if any two points in D can be joined by a path that lies in D.
3. A curve is called **closed** if its terminal point is the same as its initial point.
4. A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints.
5. A **simply-connected region** in a plane is a connected region D such that every simple closed curve in D encloses only points that are in D. No Donuts!



Looking at $\mathbf{F} = \langle 2x, 2y \rangle$ again. Can you find any functions $f(x, y)$ who have gradient $\langle 2x, 2y \rangle$? $f = x^2 + y^2$

Moreover please notice that $f(1,1) - f(0,0) = 1^2 + 1^2 - 0^2 - 0^2 = 2$

Thats to say that in our case

$$\int_C F \cdot dr = f(B) - f(A)$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_A^B \nabla f \cdot dr = f(B) - f(A)$$

Theorem 3.4 (Fundamental Theorem of Line Integrals)**.**

Let C be a smooth curve joining the point A to the point B in the plane or space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on the domain D containing C. Then:

$$\int_C F \cdot dr = f(B) - f(A)$$

equivalently:

$$\int_C \nabla f \cdot d\vec{r} = \underline{f(B)} - \underline{f(A)}$$

Theorem 3.5.

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = \underline{0}$ for every closed path C in D.

So here is where we sit:

1. We like conservative vector fields because they are path independent.
2. We like them even more because if we can find their potential function then line integrals are extremely easy to calculate.

Here are the natural questions we need to ask:

1. Given a vector field is there a way to tell if it is __conservative__?
2. Okay we know we have a conservative vector field... How can we find the __potential function__?

and very importantly we have:

Theorem 3.6 (Component Test)**.**

Let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then $\mathbf{F}$ is Conservative.

Finally! it's example time!!!!

Example 3.7. $\nabla f = \langle f_x, f_y \rangle$

Consider the vector field $\mathbf{F} = \langle e^x \cos y + y, x - e^x \sin y + 3 \rangle$.

1. Show that $\mathbf{F}$ is conservative over its natural domain

2. Find a potential function for $\mathbf{F}$.

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parametrized by $\mathbf{r}(t) = \langle t, \sin t \rangle$ and $t \in [0, 1]$

$A = (0,0) \quad B = (1, \sin(1))$

1. $\frac{\partial P}{\partial y} = P_y = -e^x \sin y + 1 \qquad Q_x = 1 - e^x \sin y$

2. $f = \int f_x \, dx = \int (e^x \cos y + y) dx = e^x \cos y + xy + c(y)$

$f_y = Q$

$\cancel{-e^x \sin y} + \cancel{x} + c'(y) = \cancel{x} \cancel{- e^x \sin y} + 3$

$c'(y) = 3$

$c(y) = 3y + K$

$f = e^x \cos y + xy + 3y + K$

3. $\int_C F \cdot dr = f(B) - f(A)$

$= e^1 \cos(\sin(1)) + \sin(1) + 3\sin(1) + \cancel{K} - [1 + \cancel{K}]$

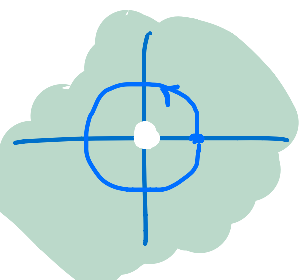
$= e\cos(\sin(1)) + 4\sin(1) - 1$

Remark 3.8 (Technique for finding potential function).

1. Integrate $\int P\,dx$ to get $f + g(y)$.
2. Try to solve for $g(y)$
   (a) Differentiate $f + g(y)$ with respect to y and set it equal to Q.
   (b) Solve for $g'(y)$.
   (c) Integrate $g'(y)$ with respect to y to get $g(y)$.

Example 3.9.

1. Show that for $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.



$$P_y = \frac{-1(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

2. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ where C is a loop parametrized by $r(t) = \langle \cos t, \sin t \rangle$, $\quad t \in [0, 2\pi]$

Solution.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \cdot \langle -\sin t, \cos t \rangle \; dt \\ &= \int_0^{2\pi} \langle \frac{-\sin t}{1}, \frac{\cos t}{1} \rangle \cdot \langle -\sin t, \cos t \rangle \; dt \\ &= \int_0^{2\pi} 1 \; dt = \boxed{2\pi}\end{aligned}$$

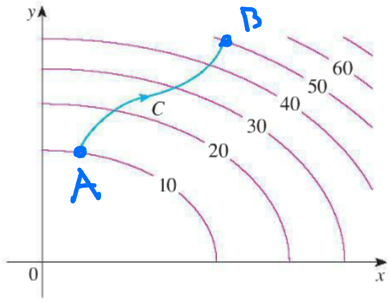
3. Does this contradict **Theorem 3.5?**

No because D is not simply connected.

Group Work

$f(B)-f(A)$
$\parallel$

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 50 - 10 = \boxed{40}$$

2. Consider the vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$

(a) Determine that $\mathbf{F}$ is conservative using the component test.

$P_y = 2x \qquad Q_x = 2x$ Yes $\vec{F}$ is conservative since $P_y = Q_x$!

(b) Find a function f such that $\mathbf{F} = \nabla f$

$$f = \int 3 + 2xy \, dx = 3x + x^2 y + g(y)$$

$$f_y = x^2 + g'(y) = x^2 - 3y^2$$

$$\Rightarrow g'(y) = -3y^2$$

$$g(y) = -y^3 + K \Longrightarrow f = \boxed{3x + x^2 y - y^3}$$

(c) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $\quad 0 \le t \le \pi$.

$A = (0,1) \longrightarrow B = (0, -e^\pi)$

$$\int_C \vec{F} \cdot d\vec{r} = f(0, -e^\pi) - f(0,1) = -(-e^\pi)^3 + (1)^3 = \boxed{e^{3\pi} + 1}$$

Note: We currently do not have the correct theory in place to show that a vector field of 3 variables is conservative. However if we assume it is conservative then we can find potential functions. Here is an example from the book (on page 1104) of how to do this for 3 variables. Please read this through if you have trouble with your WeBWorK homework assignment.

V **EXAMPLE 5** If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

$$f_x(x, y, z) = y^2 \tag{11}$$

$$f_y(x, y, z) = 2xy + e^{3z} \tag{12}$$

$$f_z(x, y, z) = 3ye^{3z} \tag{13}$$

Integrating $\boxed{11}$ with respect to x, we get

$$f(x, y, z) = xy^2 + g(y, z) \tag{14}$$

where $g(y, z)$ is a constant with respect to x. Then differentiating $\boxed{14}$ with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with $\boxed{12}$ gives

$$\underline{g_y(y, z) = e^{3z}}$$

Thus $g(y, z) = \underline{ye^{3z}} + h(z)$ and we rewrite $\boxed{14}$ as

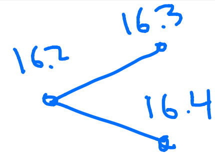
$$f(x, y, z) = \underline{xy^2} + \underline{ye^{3z}} + h(z)$$

Finally, differentiating with respect to z and comparing with $\boxed{13}$, we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

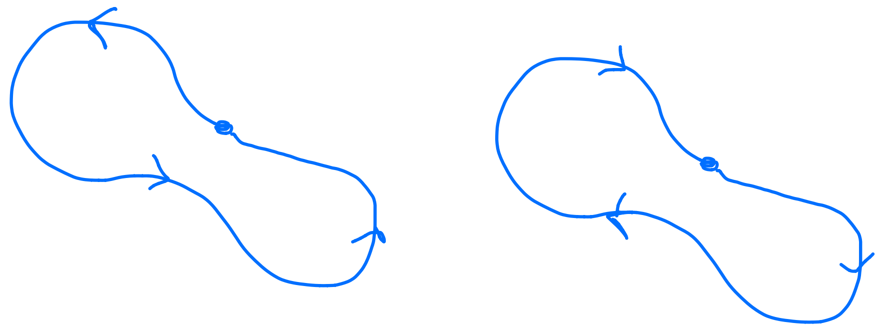
$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$.

4 Green's Theorem



Green's Theorem gives a relationship between double integrals and line integrals around simple closed curves. (Start and end at the same point. Are not self-intersecting except at endpoints.)

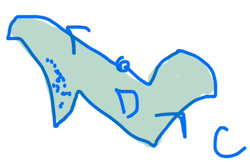


Definition(s) 4.1.

1. A simple closed curve C has positive orientation if its parametrization traverses the curve exactly once in a counterclockwise direction.
2. A simple closed curve C has negative orientation if its parametrization traverses the curve exactly once in a clockwise direction.

Theorem 4.2. Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $\mathbf{F} = \langle P, Q \rangle$ have continuous partial derivatives on an open region that contains D then,

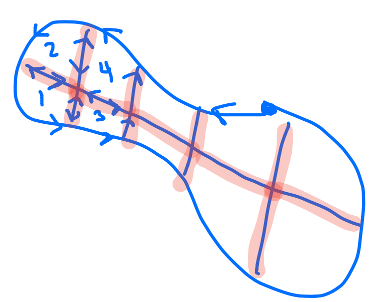


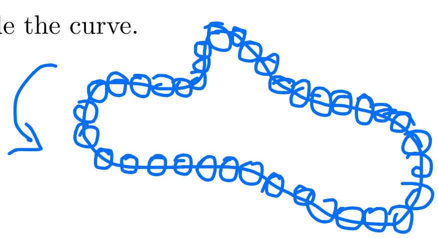
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

or equivalently

$$\int_C F \cdot T\,ds = \iint_D (Q_x - P_y)\,dA$$

The idea of the proof is important because it will come up again in Stokes' Theorem. The idea is "circulation". Because we have a closed simple curve the integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ counts how the particles on the curve are circulating. Green's Theorem says that instead of counting how the particles are circulating on the curve we can count how the particles are circulating inside the curve.

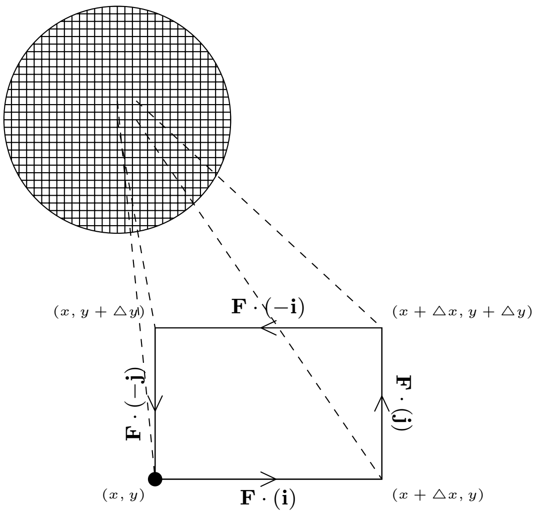




That is

$$\sum(\text{Circulation of points on curve}) = \sum(\text{Circulation of points inside curve})$$

Idea of Proof



T

$\int F \cdot T ds$

So now we need to determine circulation at a point. First lets consider circulation around small rectangles.

Along the 4 boundaries of the rectangle we get:

$$\begin{array}{llll}
\text{Top:} & \mathbf{F}(x, y+\triangle y)\cdot(-\mathbf{i})\triangle x & = & -P(x, y+\triangle y)\triangle x \\
 & \langle P,Q\rangle \cdot \langle 1,0\rangle & & \\
\text{Bottom:} & \mathbf{F}(x, y)\cdot(\mathbf{i})\triangle x & = & P(x,y)\triangle x \\
\text{Right:} & \mathbf{F}(x+\triangle x, y)\cdot\mathbf{j}\triangle y & = & Q(x+\triangle x, y)\triangle y \\
\text{Left:} & \mathbf{F}(x, y)\cdot(-\mathbf{j})\triangle y & = & -Q(x,y)\triangle y
\end{array}$$

Grouping favorably we get:

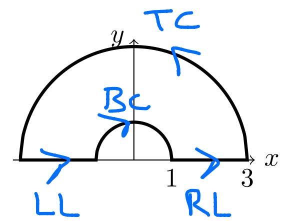
$$\begin{aligned}
\text{Circulation of } \square &= \text{Top} + \text{Bottom} + \text{Right} + \text{Left} \\
\text{Circulation of } \square &= -P(x, y+\triangle y)\triangle x + P(x,y)\triangle x + Q(x+\triangle x, y)\triangle y + -Q(x,y)\triangle y \\
\text{Circulation of } \square &= \frac{(-P(x, y+\triangle y) + P(x,y))}{\triangle y}\triangle y\triangle x + \frac{Q(x+\triangle x, y) - Q(x,y)}{\triangle x}\triangle y\triangle x \\
\text{Circulation of } \square &\approx (-P_y + Q_x)\triangle y\triangle x
\end{aligned}$$

Now we need to scale from circulation on a rectangle to circulation at a point

$$\begin{aligned}
\text{Circulation at } \bullet &\approx \frac{\text{Circulation of } \square}{\text{Area of } \square} \\
\text{Circulation at } \bullet &\approx \frac{(-P_y + Q_x)\triangle y\triangle x}{\triangle y\triangle x} = Q_x - P_y = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}
\end{aligned}$$

And so now we are ready to see why we love Green's Theorem

Example 4.3.



Find the work done by $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ once counterclockwise around the curve given by the picture:

Solution. Let's pretend we forgot Green's Theorem on the exam.

To parametrize this curve correctly I need to break it into 4 pieces

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_{BC} \mathbf{F} \cdot \mathbf{T}\ ds + \int_{TC} \mathbf{F} \cdot \mathbf{T}\ ds + \int_{LL} \mathbf{F} \cdot \mathbf{T}\ ds + \int_{RL} \mathbf{F} \cdot \mathbf{T}\ ds$$

Parametrizing the four pieces we see that (in a counterclockwise direction)

$BC:$	$\mathbf{r}(t) = \langle \cos t, \sin t \rangle$	$t \in [\pi, 0]$	$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$
$TC:$	$\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$	$t \in [0, \pi]$	$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t \rangle$
$LL:$	$\mathbf{r}(t) = \langle t, 0 \rangle$	$t \in [1, 3]$	$\mathbf{r}'(t) = \langle 1, 0 \rangle$
$RL:$	$\mathbf{r}(t) = \langle t, 0 \rangle$	$t \in [-3, -1]$	$\mathbf{r}'(t) = \langle 1, 0 \rangle$

Let's calculate these individual integrals

$$\begin{aligned}
\int_{BC} \mathbf{F} \cdot \mathbf{T}\ ds &= \int_\pi^0 \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\ dt \\
&= \int_\pi^0 \langle 4(\cos t) - 2(\sin t), 2(\cos t) - 4(\sin t) \rangle \cdot \langle -\sin t, \cos t \rangle\ dt \\
&= \int_\pi^0 -4\sin t\cos t + 2\sin^2 t + 2\cos^2 t - 4\sin t\cos t\ dt \\
&= \int_\pi^0 -8\sin t\cos t + 2\ dt \\
&= 2(0 - \pi) = -2\pi
\end{aligned}$$

$$\begin{aligned}
\int_{TC} \mathbf{F} \cdot \mathbf{T}\ ds &= \int_0^\pi \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^\pi \langle 4(3\cos t) - 2(3\sin t), 2(3\cos t) - 4(3\sin t) \rangle \cdot \langle -3\sin t, 3\cos t \rangle\ dt \\
&= \int_0^\pi -36\sin t\cos t + 18\sin^2 t + 18\cos^2 t - 36\sin t\cos t\ dt \\
&= \int_0^\pi -72\sin t\cos t + 18\ dt \\
&= 18(\pi - 0) = 18\pi
\end{aligned}$$

$$\begin{aligned}
\int_{LL} \mathbf{F} \cdot \mathbf{T}\ ds &= \int_{-3}^{-1} \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\ dt \\
&= \int_{-3}^{-1} \langle 4(t) - 2(0), 2(t) - 4(0) \rangle \cdot \langle 1, 0 \rangle\ dt \\
&= \int_{-3}^{-1} 4t\ dt \\
&= \left[2t^2\right]_{-3}^{-1} = 2(1 - 9) = -16
\end{aligned}$$

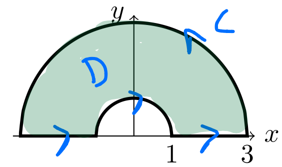
$$\begin{aligned}
\int_{RL} \mathbf{F} \cdot \mathbf{T}\ ds &= \int_1^3 \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\ dt \\
&= \int_1^3 \langle 4(t) - 2(0), 2(t) - 4(0) \rangle \cdot \langle 1, 0 \rangle\ dt \\
&= \int_1^3 4t\ dt \\
&= \left[2t^2\right]_1^3 = 2(9 - 1) = 16
\end{aligned}$$

Giving us our final answer of

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = -2\pi + 18\pi - 16 + 16 = \boxed{16\pi}$$

Now let's imagine you remember Green's Theorem.

Example 4.3.



Find the work done by $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ once counterclockwise around the curve given by the picture:

$$\text{Work} = \int_C F \cdot T\, ds = \iint_D (Q_x - P_y)\, dA$$

$$= \iint_D 2 + 2 \quad dA$$

$$= \iint_D 4\, dA$$

$$= \int_0^{\pi} \int_1^3 4r\, dr\, d\theta$$

$$= \pi \left[2r^2 \right]_1^3 = \pi [18 - 2]$$

$$= \boxed{16\pi}$$

Notation 4.4.

1. The notation

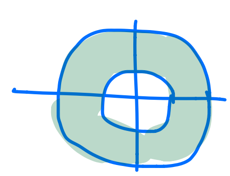
$$\oint_C P\,dx + Q\,dy$$

Is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

2. Another notation for the positively oriented boundary curve of a region D is $\underline{\partial D}$.

Fun Reads

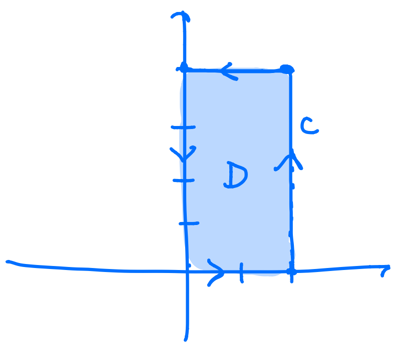
There is additional material in 16.4 that is covered in the book that MSU will not currently be testing on. Those wishing to gain a greater understanding of the power of Green's Theorem may wish to read the section on finding area using line integrals (top of page 1111) and the section on **Extended Versions of Green's Theorem** (starting on page 1111).



$\iint Q_x - P_y \, dA$

Group Work

1. (WW#3) Use Green's Theorem to evaluate the line integral $\oint_C \underbrace{4\cos(-y)}_{P}\ dx + \underbrace{4x^2\sin(-y)}_{Q}\ dy$. Where C is the rectangle with vertices $(0,0), (2,0), (0,4)$, and $(2,4)$.



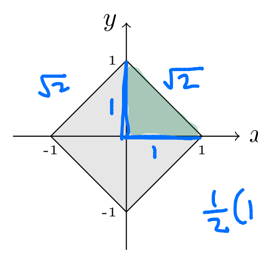
$$\iint_D 8x\sin(-y) - 4\sin(-y)\ dA$$

$$= \int_0^2 \int_0^4 (8x-4)\sin(-y)\ dy\, dx$$

$$= \left[4x^2 - 4x\right]_0^2 \left[\cos(-y)\right]_0^4$$

$$\boxed{8\left[\cos(-4) - 1\right]}$$

2. Calculate $\oint_C (x^4 + 2y)dx + (5x + \sin y)dy$ where C is the boundary of region shown to the right:



$\frac{1}{2}(1)(1) = \frac{1}{2}$

$$\iint_D 5 - 2\, dA = 3\iint_D dA = 3(\sqrt{2})^2 = \boxed{6}$$

5 Curl and Divergence

Before we get to Curl and Divergence we need a new operator.

Definition(s) 5.1.

The vector differential operator ∇ (pronounced "del") is defined as:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Definition(s) 5.2.

The curl of **F** is the vector field on $\mathbb{R}^3$ defined by:

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \vec{F} \\
&= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\
&= (R_y - Q_z)i - (R_x - P_z)j + (Q_x - P_y)k \\
&= \langle R_y - Q_z, P_z - R_x, \underline{Q_x - P_y} \rangle
\end{aligned}$$

Note: Does third component look familiar? It 2 dimensional "circulation at a point" that we integrated in Green's Theorem. Also if $Q_x - P_y = 0 \iff Q_x = P_y$ which is the major condition in the component test.

Theorem 5.3.

If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $F = \vec{0}$, then **F** is conservative.

Example 5.4.

$$f_x \quad f_y \quad f_z$$
$$\parallel \quad \parallel \quad \parallel$$

Consider the vector field $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$.

(a) Show that $\mathbf{F}$ is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)i - (3y^2z^2 - 3y^2z^2)j + (2yz^3 - 2yz^3)k$$

$$= \langle 0, 0, 0 \rangle$$

$$f = \int y^2z^3\,dx = \underline{xy^2z^3 + g(y,z)}$$

$$f_y = \cancel{2xyz^3} + g_y(y,z) = \cancel{2xyz^3}$$

$$g_y(y,z) = 0$$

$$g(y,z) = g(z)$$

$$f_z = \cancel{3xy^2z^2} + g'(z) = \cancel{3xy^2z^2}$$

$$g'(z) = 0$$

$$g(z) = K$$

$$f = \boxed{xy^2z^3 + K}$$

Definition(s) 5.5.

The divergence of **F** is scalar function defined by:

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot \vec{F} \\ &= \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle \\ &= P_x + Q_y + R_z \end{aligned}$$

Example 5.6.

If $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$ find div **F**.

$\text{div } F = z + xz + 0$

Great but why is this useful?

Theorem 5.7.

If $\mathbf{F} = \langle P, Q, R \rangle$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div}(\text{curl } F) = 0$$

Note: Proof of this is in the book on page 1119. It is very boring and not at all enlightening. It works because of Clairaut's Theorem.

Example 5.8.

Your friend Eugene comes up to you and is like "*Whoa you have to check out my awesome vector field* ***F***. *You know what? I bet you can't even figure out what it is. The only thing I'll tell you is that curl* $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$." Shut Eugene up by finding his vector field if it exists or prove that Eugene is a liar.

$R_y - Q_z = xz$, $P_z - R_x = xyz$, $Q_x - P_y = -y^2$

$$\begin{aligned} \text{div}(\text{curl } F) &= \text{div}(\langle xz, xyz, -y^2 \rangle) \\ &= z + xz \neq 0 \end{aligned}$$

Eugene is a liar!

Definition(s) 5.9.

f

1. Curl helps to measure rotations about a point. Because of this if curl **F**= 0 at a point P then the fluid is ree from rotations at P and **F** is called irrotational at P.

2. Div (Divergence) represents the net rate of change (with respect to time) of a mass of fluid (or gas) flowing from the point (x, y, z). If div **F**= 0 then there is no net change and **F** is said to be incompressible.

Alternate Forms of Green's Theorem

$$\text{curl}\left(\langle y, -x, 0\rangle\right) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = (0)i - (0)j + (-1-1)k = \langle 0, 0, -2\rangle$$

Two ideas here that will be used later. Both have to do with downgrading curl and divergence to 2 dimensions for a minute. That is take $\mathbf{F} = \langle P, Q, 0\rangle$

Theorem 5.10 (Green's Thoerem)**.**

Bunch of conditions up here.

$$\langle -, -, Q_x - P_y\rangle \cdot \langle 0, 0, 1\rangle$$

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

$$= \iint_D \text{curl}\, \vec{F} \cdot \vec{k}\, dA$$

And while this may be annoying to write right now, it is the first good step in expanding Green's Theorem to 3 dimensions and discovering Stokes' Theorem (16.8).

The second idea is instead of choosing to do the line integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ instead evaluating $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ where $\mathbf{n}$ is a outward pointing unit normal vector vector to C.

Theorem 5.11.

An outward pointing unit normal vector to a curve C parametrized counterclockwise by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is given by:

$$n(t) = \frac{1}{|r'(t)|}\langle y'(t), -x'(t)\rangle$$

Theorem 5.12 (Green's Thoerem Alternate)**.**

Bunch of conditions up here.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\,dA$$

$$= \iint_D \operatorname{div} F \, dA$$

A proof of this can be found on page 1120. This will help us expand into 3 dimensions for Divergence Theorem (16.9).

Example 5.13.

Let $\mathbf{F}=-2x\mathbf{i}-3y\mathbf{j}+5z\mathbf{k}$. Is $\mathbf{F}$ irrotational/incompressible/both/neither?

$\operatorname{div} F = -2-3+5=0$

yes, incompressible

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x & -3y & 5z \end{vmatrix}$$

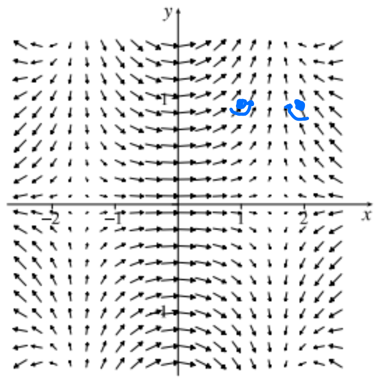
$= (\,0\,)i - (\,0\,)j + (\,0\,)k$

$= \langle 0,0,0\rangle$

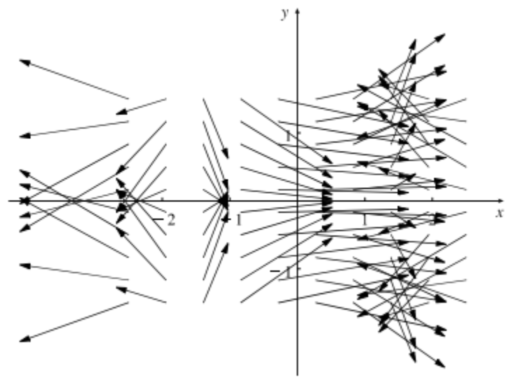
yes; irrotational

Group Work

1. Consider the vector field $\mathbf{F}(x, y) = \langle 2\cos x, \sin(xy) \rangle$ shown below.



Scaled



Unscaled

(a) Find formulas for divergence and curl of $\mathbf{F}$.

$\operatorname{div} \vec{F} = \underline{-2\sin x + \cos(xy)\cdot(x)}$

$\operatorname{curl} F = \langle 0, 0, y\cos(xy) \rangle$

(b) Show that the divergence is 0 everywhere along the $y-$axis. How is this apparent in the graph?

$x = 0 \Rightarrow -2\sin(0) + \cos(0)\cdot(0) = 0$

the vectors entering into the y-axis have approximately the same magnitude as the vectors leaving the y-axis.

(c) Find the curl at $\left(\frac{\pi}{3}, 1\right)$ and $\left(\frac{2\pi}{3}, 1\right)$. Relate the sign difference in your answer to the direction of the curl.

$\langle 0, 0, \cos(\frac{\pi}{3}) \rangle = \langle 0, 0, \frac{1}{2} \rangle$

$\langle 0, 0, \cos(2\frac{\pi}{3}) \rangle = \langle 0, 0, -\frac{1}{2} \rangle$

implicit $F(x,y,z)=c$

$x^2+y^2+z^2=1$

6 Surface Area

Just as arc length is an application of a single integral, surface area is an application of double integrals.

- In 15.6 we compute surface area for explicit surfaces $z=f(x,y)$. In chapter 16 we compute surface area of parametrized surfaces.

Arc length: $L(C)=\int_a^b\sqrt{\left(\frac{dy}{dx}\right)^2+1}\,dx$

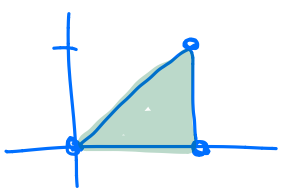
Theorem 6.1.

The area of the surface with equation $z=f(x,y)$ with $(x,y)\in D$, where f_x and f_y are continuous, is:

$$A(S)=\iint_D\sqrt{(f_x)^2+(f_y)^2+1}\;dA$$

Example 6.2.

Find the surface area of the part of the surface $z=x^2+2y$ that lies above the triangular region T in the xy- plane with vertices $(0,0),(1,0)$, and $(1,1)$.



$f(x,y)=x^2+2y \qquad f_x=2x,\quad f_y=2$

$$A(S)=\iint_D\sqrt{4x^2+4+1}\,dA$$

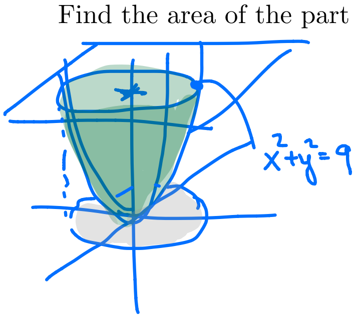
$$=\int_0^1\int_0^x\sqrt{4x^2+5}\,dy\,dx$$

$$=\int_0^1 x\sqrt{4x^2+5}\,dx=\frac{2}{3}\frac{1}{8}(4x^2+5)^{3/2}\Big|_0^1$$

$$=\boxed{\frac{1}{12}\left[27-5^{3/2}\right]}$$

Example 6.3.

Find the area of the part of the paraboloid $z=x^2+y^2$ that lies under the plane $z=9$.



$f(x,y)=x^2+y^2 \qquad f_x=2x,\quad f_y=2y$

$$A(S)=\iint_D\sqrt{4x^2+4y^2+1}\;dA$$

$$=\int_0^{2\pi}\int_0^3\sqrt{4r^2+1}\;r\,dr\,d\theta$$

$$=2\pi\left[\frac{2}{3}\frac{1}{8}(4r^2+1)^{3/2}\right]_0^3$$

$$=\boxed{\frac{\pi}{6}\left[37^{3/2}-1\right]}$$

6 Parametric Surfaces and Their Areas

$X+y+z=1$

Take time to read this and watch the videos before coming to class.

Recall that curves in space are 1 dimensional so we need 1 dimensional $\implies$ 3 dimensional so they look like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Recall that surfaces in space are 2 dimensional so we need 2 dimensional $\implies$ 3 dimensional so they look like $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

In the book they have a habit of instead writing:

$$\mathbf{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$$

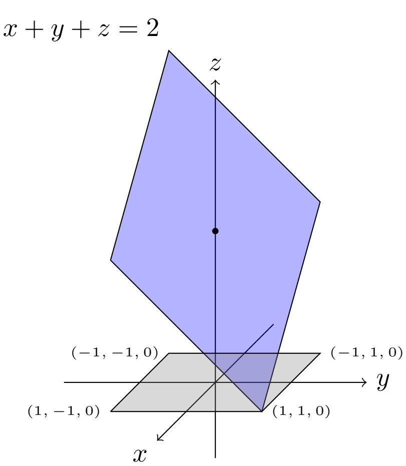
To get a better idea of the visualization and mathematics behind surface parametrization please take advantage of the following videos (they are hyper links so just click on them on your computer). The videos take about 35min to go through but you will have a much better understanding of this subject. I highly recommend it.

Video 1:
https://www.khanacademy.org/math/calculus/multivariable-calculus/surface_parametrization/v/introduction-to-parametrizing-a-surface-with-two-parameters

Video 2:
https://www.khanacademy.org/math/calculus/multivariable-calculus/surface_parametrization/v/determining-a-position-vector-valued-function-for-a-parametrization-of-two-parameters

Example 6.1.

Give a parametrization $\mathbf{r}(s,t)$ of the plane $x + y + z = 2$ over the square $x \in [-1,1]$ and $y \in [-1,1]$. (Picture to right)



$x = s$

$y = t$

$z = 2 - s - t$

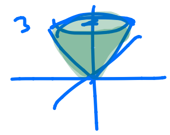
$r(s,t) = \langle s, t, 2-s-t \rangle$

$s \in [-1, 1]$ and $t \in [-1, 1]$

Example 6.2.

$z = 3r$

Find a parametrization $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ of the following: $z = 3\sqrt{x^2+y^2}$ $\quad z \in [0,3]$



$\cancel{x = s}$

$\cancel{y = t}$

$\cancel{z = 3\sqrt{s^2+t^2}}$

$\cancel{r(s,t) = \langle s, t, 3\sqrt{s^2+t^2}\rangle}$

$x = r\cos\theta$

$x = s\cos t$

$y = s\sin t$

$z = 3\sqrt{s^2\cos^2 t + s^2\sin^2 t}$

$= 3s$

$r(s,t) = \langle s\cos t, s\sin t, 3s\rangle$

$s \in [0,1]$ and $t \in [0, 2\pi]$

Example 6.3.

Find a parametrization $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ of the following: $x^2+y^2+z^2 = 9$

$x = \rho\cos\theta\sin\phi$

$x = 3\cos\theta\sin\phi$

$x = 3\cos t\sin s$

$y = 3\sin t\sin s$

$z = 3\cos s$

$r(s,t) = \langle 3\cos t\sin s, 3\sin t\sin s, 3\cos s\rangle$

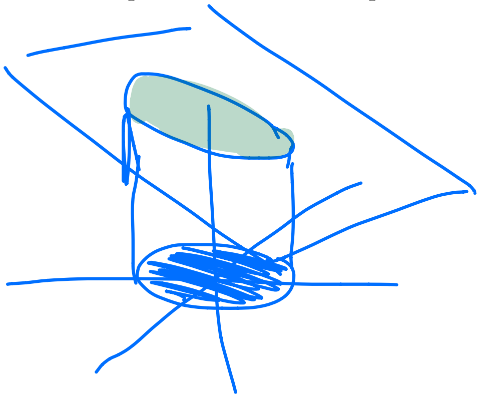
$s \in [0,\pi]$ and $t \in [0, 2\pi]$

Remark 6.4.

If you are doing things right then often your limits for your independent variables s, t (or u, v) should not depend on one another.

Example 6.5.

Parametrize the portion of the tilted plane $x - y + 3z = 5$ that lies inside of the cylinder $x^2 + y^2 = 4$.



$\cancel{r(s,t) = \langle s, t, \frac{5+t-s}{3} \rangle}$

$\cancel{s \in [\ ,\ ] \text{ and } t \in [\ ,\ ]}$

$x = s\cos t$

$y = s\sin t$

$z = \dfrac{5 - x + y}{3}$

$= \dfrac{5 - s\cos t + s\sin t}{3}$

$r(s,t) = \left\langle s\cos t, s\sin t, \dfrac{5 - s\cos t + s\sin t}{3} \right\rangle$

$s \in [0, 2]$ and $t \in [0, 2\pi]$

Remark 6.6.

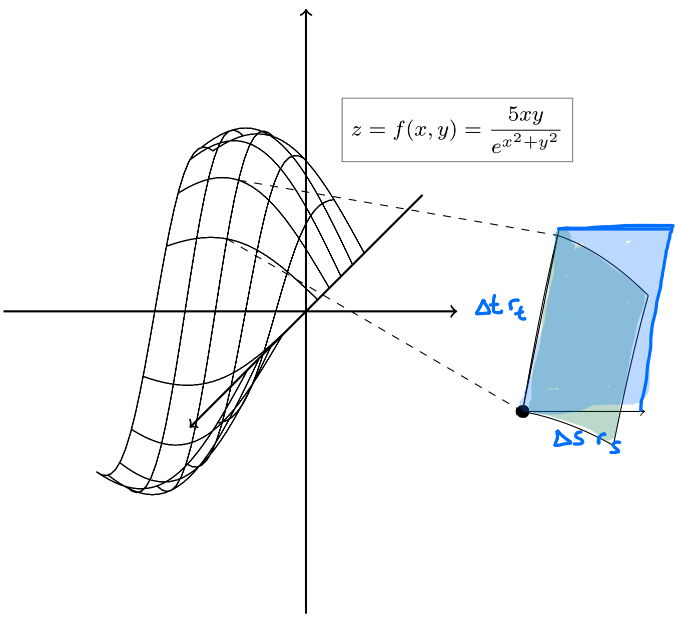
Compare this problem to 6.1. These two problems together should teach us that it isn't just about the surface, it's also the area we the surface is over.

Definition(s) 6.7.

A parametrized surface $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$ is smooth if r_s and r_t are continuous and $r_s \times r_t$ is never zero on the interior of the parameter domain.

Note: our parametrization of the cone is smooth.

Now our goal is to find and area equation for smooth parametrized surfaces. The idea is as follows



If we chop our surface into lots of small rectangles that have side lengths Δs and Δt (where s, t parametrize the surface) then the area of the small surface piece is about the area of the parallelogram made by the vectors $\Delta s\, r_s$ and $\Delta t\, r_t$

Then we can sum all these areas together to get the summation:

$$\sum_n \text{area of //-ogram}$$
$$= \sum_n |\Delta s r_s \times \Delta t\, r_t|$$
$$= \sum_n |r_s \times r_t|\, \Delta s\, \Delta t$$

Since $|\mathbf{r}_s \times \mathbf{r}_t|$ is continuous we know that

$$\text{Surface Area } = \sum_n |\mathbf{r}_s \times \mathbf{r}_t| \triangle t\ \triangle s = \iint_D |r_s \times r_t|\, ds\, dt$$

Theorem 6.8.

The **area** of a smooth surface $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ with $s \in [a,b]$ and $t \in [c,d]$ is:

$$\text{Area } = \int_c^d \int_a^b |r_s \times r_t|\, ds\, dt$$

Or the book writes:

Theorem 6.9.

The **area** of a smooth surface $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ with $a \le u \le b$ and $c \le v \le d$ is:

$$\text{Area } = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

Remark 6.10. Note we can use this theorem along with the parametrization $\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$ to prove the 15.6 formula for surface area.

There is still much to cover so lets talk about the following worked out example:

Example 6.11.

Find the area of the portion of the tilted plane $x - y + 3z = 5$ that lies inside of the cylinder $x^2 + y^2 = 4$.

Solution. From Example 5.2 we have that the tilted plane is parametrized by $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, \frac{5 + s\sin t - s\cos t}{3}\rangle$ where $s \in [0,2]$ and $t \in [0, 2\pi]$

So we first need to calculate out:

$$\begin{aligned} \mathbf{r}_s &= \langle \cos t, \sin t, \frac{\sin t - \cos t}{3}\rangle \\ \mathbf{r}_t &= \langle -s\sin t, s\cos t, \frac{s\cos t + s\sin t}{3}\rangle \end{aligned}$$

Now we can calculate $|\mathbf{r}_s \times \mathbf{r}_t|$

$$\begin{aligned} |\mathbf{r}_s \times \mathbf{r}_t| &= \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & \frac{\sin t - \cos t}{3} \\ -s\sin t & s\cos t & \frac{s\cos t + s\sin t}{3} \end{matrix} \right\| \\ &= \left|(\sin t\frac{s\cos t + s\sin t}{3} - s\cos t\frac{\sin t - \cos t}{3})\mathbf{i} - (\cos t\frac{s\cos t + s\sin t}{3} + s\sin t\frac{\sin t - \cos t}{3})\mathbf{j} + (s\cos^2 t + s\sin^2 t)\mathbf{k}\right| \\ &= \left|(\frac{s\sin\cos t + s\sin^2 t}{3} + \frac{-s\sin t\cos t + s\cos^2 t}{3})\mathbf{i} - (\frac{s\cos^2 t + s\sin t\cos t}{3} + \frac{s\sin^2 t - s\sin t\cos t}{3})\mathbf{j} + (s)\mathbf{k}\right| \\ &= \left|(\frac{s\sin^2 t}{3} + \frac{s\cos^2 t}{3})\mathbf{i} - (\frac{s\cos^2 t}{3} + \frac{s\sin^2 t}{3})\mathbf{j} + (s)\mathbf{k}\right| \\ &= \left|(\frac{s}{3})\mathbf{i} - (\frac{s}{3})\mathbf{j} + (s)\mathbf{k}\right| \\ &= \sqrt{\left(\frac{s}{3}\right)^2 + \left(\frac{-s}{3}\right)^2 + s^2} \\ &= \sqrt{\frac{s^2}{9} + \frac{s^2}{9} + \frac{9s^2}{9}} = \sqrt{\frac{11s^2}{9}} = \boxed{\frac{s}{3}\sqrt{11}} \end{aligned}$$

Integrating finally we get:

$$\begin{aligned}
\int_0^{2\pi}\int_0^2 |\mathbf{r}_s \times \mathbf{r}_t| \; ds \; dt &= \int_0^{2\pi}\int_0^2 \frac{s}{3}\sqrt{11} \; ds \; dt \\
&= \left[\int_0^{2\pi} 1 \; dt\right]\left[\int_0^2 \frac{s}{3}\sqrt{11} \; ds\right] \\
&= [2\pi]\left[\frac{s^2}{6}\sqrt{11}\right]_0^2 \\
&= 2\pi\frac{4}{6}\sqrt{11} \\
&= \boxed{\frac{4\pi}{3}\sqrt{11}}
\end{aligned}$$

Intuitively a slanted circle like this should have more area than a non-slanted circle in the cylinder so we could check:

$$\pi(2^2) = 4\pi < \frac{4\pi}{3}\sqrt{11}$$

Since $\sqrt{11} > 3$. So our answer seems reasonable!

Remember: unless the problem specifies you can use 15.6 to help your evaluation.

Example 6.12.

Find the area of the portion of the tilted plane $z = \dfrac{5-x+y}{3}$ (Look familiar?... $x - y + 3z = 5$) that lies inside of the cylinder $x^2 + y^2 = 4$

Solution.

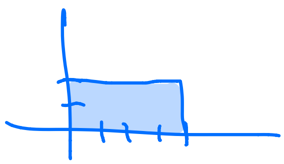
$$\begin{aligned}
z &= \frac{5-x+y}{3} \\
f(x,y) &= \frac{5-x+y}{3} \\
f_x(x,y) &= \frac{-1}{3} \\
f_y(x,y) &= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Area} &= \iint_R \sqrt{f_x^2 + f_y^2 + 1} dy \; dx \\
&= \iint_R \sqrt{\frac{1}{9} + \frac{1}{9} + 1} dy \; dx \\
&= \iint_R \sqrt{\frac{11}{9}} dy \; dx \\
&= \frac{\sqrt{11}}{3}\iint_R dy \; dx \\
&= \frac{\sqrt{11}}{3}(\pi 2^2) = \boxed{\frac{4\pi\sqrt{11}}{3}};
\end{aligned}$$

Group Work

Note: $\int \frac{1}{\sqrt{a-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)$

1. Find the area of the part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices $(0,0), (4,0), (0,2)$, and $(4,2)$.



$f(x,y) = z = \sqrt{9-y^2}$

$f_x = 0, \quad f_y = \frac{1}{\cancel{2}}(9-y^2)^{-1/2}(-\cancel{2}y) = \frac{-y}{\sqrt{9-y^2}}$

$$\text{Area} = \iint \sqrt{0^2 + \frac{y^2}{9-y^2} + 1}\, dA = \int_0^4 \int_0^2 \sqrt{\frac{y^2 + (9-y^2)}{9-y^2}}\, dy\, dx$$

$$= 4\int_0^2 \frac{3}{\sqrt{9-y^2}}\, dy = 4\left[3\sin^{-1}\left(\frac{x}{3}\right)\right]_0^2$$

$$= 12\left[\sin^{-1}\left(\tfrac{2}{3}\right) - \sin^{-1}(0)\right]$$

$$= 12\sin^{-1}(2/3)$$

2. Express that area of the surface $z = e^{-x^2-y^2}$ that lies above the disk $x^2 + y^2 \leq 4$ in terms of a single integral. Do not evaluate.

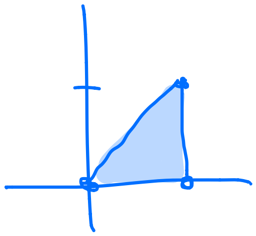
$f(x,y) = e^{-x^2-y^2} \qquad f_x = -2xe^{-x^2-y^2} \qquad f_y = -2ye^{-x^2-y^2}$

$$\text{Area} = \iint_D \sqrt{4x^2e^{-2(x^2+y^2)} + 4y^2e^{-2(x^2+y^2)} + 1}\, dA$$

$$= \int_0^{\pi}\int_0^2 \sqrt{4r^2e^{-2r^2} + 1}\, r\,dr\,d\theta = \pi\int_0^2 r\sqrt{4r^2e^{-2r^2}+1}\, dr$$

3. Give a parametrization $\mathbf{r}(s,t)$ of the surface $z = xy^2$ over the triangle with vertices $(0,0), (1,0)$, and $(1,1)$ in the xy–plane.

$x = s, \quad y = t \quad z = st^2$



$r(s,t) = \langle s, t, st^2 \rangle \quad s \in [0,1], \; t \in [0,s]$

4. Use this parametrization to express the surface area as a double integral. Simplify as much as possible without evaluating the integrals.

$r_s = \langle 1, 0, t^2 \rangle \quad r_t = \langle 0, 1, 2st \rangle$

$$r_s \times r_t = \begin{vmatrix} i & j & k \\ 1 & 0 & t^2 \\ 0 & 1 & 2st \end{vmatrix} = \langle -t^2, -2st, 1 \rangle$$

$$|r_s \times r_t| = \sqrt{t^4 + 4s^2t^2 + 1}$$

$$\text{Area} = \iint_D |r_s \times r_t| \, dt\, ds = \int_0^1 \int_0^s \sqrt{t^4 + 4s^2t^2 + 1} \; dt\, ds$$

7 Surface Integrals

Class Learning Goals

1. Calculate surface integrals of scalar functions
   (a) Given an explicit surface, $z = g(x, y)$.
   (b) Given a parametric surface, $\mathbf{r}(s, t)$.
2. Gain an intuitive understanding of an oriented surface with orientation given by the unit normal vector and the concept of positive orientation.

In 16.2 we upgraded from finding arc length, $\int_C 1\, ds$ to finding line integrals of scalar functions $\int_C f(x, y, z) ds$ where that $ds =$ $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+1}$. We will now upgrade from surface area to surface integrals.

Definition(s) 7.1.

1. If a surface S is given explicitly as $z = g(x, y)$ then define $dS =$ $\sqrt{(g_x)^2+(g_y)^2+1}\; dy\, dx$
2. If a surface S is given parametrically as $\mathbf{r}(s, t)$ then define $dS =$ $|r_s \times r_t|\; ds\, dt$

Bonus Exercise: Show that explicit surfaces can be parametrized $x = s$, $y = t$, and $z = g(s, t)$. When done so then $|\mathbf{r}_s \times \mathbf{r}_t| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$.

Remark 7.2.

With this definition surface area of a surface S can be expressed as $\iint_S 1\, dS$.

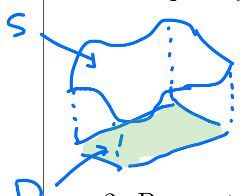
Definition(s) 7.3.

The surface integrals **of f over the surface** S is given by

$$\iint_S f(x,y,z)\,dS$$

which can be expressed

1. Explicitly:



$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2+(g_y)^2+1}\,dy\,dx$$

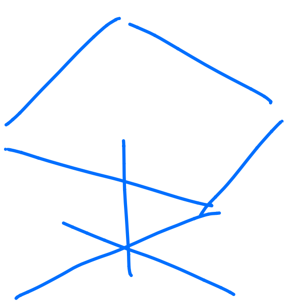
2. Parametrically:

$$\iint_S f\,dS = \iint_D f(\vec{r}(s,t))\,|r_s \times r_t|\,ds\,dt$$

Example 7.4 (FS14 Exam 4 Question).

$z = 4-2x-y$
$g(x,y) = 4-2x-y$

Consider the entire plane $2x+y+z=4$ which I have loaded with snakes according to the snake density function $N(x,y,z) = \dfrac{50}{\pi\sqrt{6}}e^{-x^2-y^2}\dfrac{snakes}{m^2}$. How many snakes are on the plane?



$$\iint_S N(x,y,z)\,dS = \iint_D \frac{50}{\pi\cancel{\sqrt{6}}} e^{-x^2-y^2}\cancel{\sqrt{4+1+1}}\,dy\,dx$$

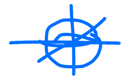
$$= \int_0^{2\pi}\int_0^{\infty} \frac{50}{\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$= 2\pi\,\frac{50}{\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}$$

$$= 100\left[0+\frac{1}{2}\right] = 50 \text{ snakes}$$

Example 7.5. Compute the surface integral $\iint_S z^2 \, dS$, where S is the unit sphere $x^2+y^2+z^2=1$.

(a) by expressing the surface parametrically.



$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$dz\,dy\,dx = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$r(s,t) = \langle \cos t \sin s, \sin t \sin s, \cos s \rangle$$

$s \in [0, \pi]$

$t \in (0, 2\pi]$

$$r_s = \langle \cos t \cos s, \sin t \cos s, -\sin s \rangle$$

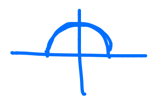
$$r_t = \langle -\sin t \sin s, \cos t \sin s, 0 \rangle$$

$$r_s \times r_t = \langle \cos t \sin^2 s, \sin t \sin^2 s, \cos s \sin s \rangle$$

$$|r_s \times r_t| = \sqrt{\cos^2 t \sin^4 s + \sin^2 t \sin^4 s + \cos^2 s \sin^2 s}$$

$$= \sqrt{\sin^4 s + \cos^2 s \sin^2 s}$$

$$= \sqrt{\sin^2 s (1)} = \sin s$$

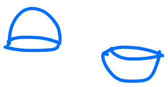


$$\iint_S z^2 \, dS = \int_0^{2\pi}\int_0^{\pi} \cos^2 s \sin s \, ds \, dt$$

$$= 2\pi \left[-\frac{1}{3}\cos^3 s \right]_0^{\pi}$$

$$= 2\pi \left[\frac{1}{3} + \frac{1}{3}(1) \right] = \boxed{\frac{4\pi}{3}}$$

(b) by carefully ripping the surface into two surfaces that can be expresses explicitly.



$$x^2+y^2+z^2=1$$

$$z=\pm\sqrt{1-x^2-y^2}$$

Top $\quad z=g(x,y)=\sqrt{1-x^2-y^2}$

$$\iint_{S_T} z^2\,dS_T = \iint_D (1-x^2-y^2)\sqrt{1+\left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2}\,dy\,dx$$

$$=\iint_D \sqrt{(1-x^2-y^2)}^2\sqrt{\frac{1-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{1-x^2-y^2}}\,dy\,dx$$

$$=\iint_D \sqrt{1-x^2-y^2}\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\theta$$

$$=2\pi\left[\frac{-1}{\cancel{2}}\frac{\cancel{2}}{3}(1-r^2)^{3/2}\right]_0^1$$

$$=2\pi\left[\frac{1}{3}\right]=\frac{2\pi}{3}$$

$$\text{Bottom} \rightarrow \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$$

Top

We are moving into defining surface integrals of vector fields (just as in 16.2 after line integrals of scalar functions we moved into line integrals of vector fields). To do this correctly we need a sense of orientation.

The book goes into a very formal definition for orientation. However non orientable surfaces are quite rare (extinct in this class). We will just settle with an intuitive definition

Definition(s) 7.6 (ish)**.**

1. A surface S is called an Oriented surface if it has two sides.
   EX: Planes, Sphere, All quadric surfaces
   EX: Non oriented surfaces: Mobius Strip, Klein Bottle
2. An orientation is just a choosing of one of the two sides. Some common orientations include:
   (a) Outward (for closed surfaces [see below])
   (b) Upward has a positive z-component.
3. In this class choosing an orientation comes down to selecting the correct direction.

One of our remain big theorems, **the divergence theorem**, will depend on closed surfaces so we will give it a few extra definitions.

Definition(s) 7.7.

1. A surface is called **closed** if it is the boundary of a solid region.
2. Outward orientation is also referred to as positive orientation.

Insert Ryan rant about why orientation is necessary for the upcoming material.

Group Work

1. Evaluate $\iint_S y\ dS$, where S is the surface $z = x + y^2$, $\quad x \in [0,1]$, $\quad y \in [0,2]$.

$$dS = \sqrt{(1)^2 + (2y)^2 + 1}\ dA$$

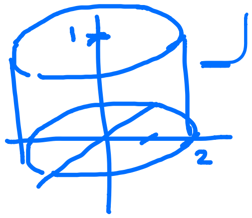
$$\iint_S y\, dS = \iint_D y\sqrt{2+4y^2}\ dA = \int_0^1 \int_0^2 y\sqrt{2+4y^2}\ dy\, dx$$

$$= 1\left[\frac{2}{3}\frac{1}{8}\left(2+4y^2\right)^{3/2}\right]_0^2$$

$$= \boxed{\frac{1}{12}\left[18^{3/2} - 2\right]}$$

2. Evaluate: $\displaystyle\iint_S (x^2+y^2+z^2)\, dS$

where is is the part of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=1$, together with its top and bottom disks.



$S = T \cup B \cup S_1$

top, bottom, sides

On 1

$$\iint_T (x^2+y^2+1)\sqrt{0^2+0^2+1}\, dA = \int_0^{2\pi}\int_0^2 (r^2+1)\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^2 = 2\pi[4+2]$$

$$= 12\pi$$

On B

$$\iint_B (x^2+y^2+0)\sqrt{0^2+0^2+1}\, dA = \int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{r^4}{4}\right]_0^2 = 8\pi$$

On S_1

$r(s,t) = \langle 2\cos t, 2\sin t, s\rangle$

$s\in[0,1] \quad t\in[0,2\pi]$

$r_s = \langle 0,0,1\rangle$

$r_t = \langle -2\sin t, 2\cos t, 0\rangle$

$$r_s\times r_t = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -2\sin t & 2\cos t & 0\end{vmatrix} = \langle 2\cos t, -2\sin t, 0\rangle$$

$$|r_s\times r_t| = \sqrt{4\cos^2 t+4\sin^2 t+0^2} = 2$$

$$\iint_S x^2+y^2+z^2\, dS = \int_0^{2\pi}\int_0^1 (4+s^2)(2\,ds\,dt)$$

$$= 2\pi(2)\left[4s+\frac{s^3}{3}\right]_0^1$$

$$= 4\pi\left[4+\frac{1}{3}\right] = \frac{52\pi}{3}$$

$$\iint_S x^2+y^2+z^2\, dS = \iint_T x^2+y^2+z^2\, dT + \iint_B x^2+y^2+z^2\, dB + \iint_{S_1} x^2+y^2+z^2\, dS_1$$

$$= 12\pi + 8\pi + \frac{52\pi}{3} = \frac{112\pi}{3}$$

7 Surface Integrals

Class Learning Goals

1. Calculate surface integrals of vector fields.
   (a) Given an explicit surface, $z = g(x, y)$.
   (b) Given a parametric surface, $\mathbf{r}(s, t)$.
2. Recognize the physical interpretation of the above calculations.

Now we are ready to transition into surface integrals of vector fields.

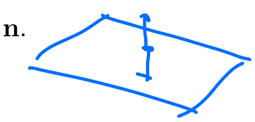
Definition(s) 7.8.

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the Surface integral **of F over** S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

This integral is also called the flux of $\mathbf{F}$ across S.

To help evaluate these we need to determine a more concrete formula for $\mathbf{n}$.



1. If the surface is defined parametrically by $\mathbf{r}(s, t)$, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S F \cdot \frac{\pm(r_s \times r_t)}{|r_s \times r_t|}\, dS$$

$|r_s \times r_t|\, ds\, dt$

$$= \iint_D F \cdot \pm(r_s \times r_t)\, ds\, dt$$

2. If the surface is defined explicitly by $z = g(x, y)$, then

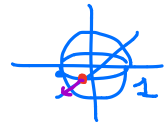
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D F \cdot \frac{\pm\langle g_x, g_y, -1\rangle}{\cancel{\sqrt{g_x^2+g_y^2+1}}} \cancel{\sqrt{g_x^2+g_y^2+1}}\, dy\, dx = \iint_D F \cdot \pm\langle g_x, g_y, -1\rangle\, dA$$

2 Notes: The $\mathbf{n}$ in the explicit equation should look familiar from 14.4 (Tangent Planes). All these $\pm$ signs are determined by the surface's orientation.

$g_x(x-x_0)+g_y(y-y_0)-(z-z_0)=0$

$z=g(x,y)$

Example 7.9.



Find the (outward) flux of vector field $\mathbf{F} = \langle z, y, x\rangle$ across the sphere parametrized by $\mathbf{r}(s,t) = \langle \sin s\cos t, \sin s\sin t, \cos s\rangle$ with $s \in [0,\pi]$, $t \in [0, 2\pi]$. **Hints:** Recall $\mathbf{r}_s \times \mathbf{r}_t = \langle \sin^2 s\cos t, \sin^2 s\sin t, \sin s\cos s\rangle$, $\int_0^{\pi} \sin^3 u\, du = 4/3$, and $\int_0^{2\pi} \sin^2 u\, du = \pi$.

$$\text{Flux} = \iint_S F\cdot d\vec{S} = \iint_D F\cdot \pm(r_s\times r_t)\,ds\,dt$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle z,y,x\rangle \cdot \pm\langle \sin^2 s\cos t, \sin^2 s\sin t, \sin s\cos s\rangle\,ds\,dt$$

$s = \pi/2 \quad t = 0$

$r_s \times r_t = \langle 1, 0, 0\rangle$

$$= \int_0^{2\pi}\int_0^{\pi} \langle \cos s, \sin s\sin t, \sin s\cos t\rangle \cdot \langle \sin^2 s\cos t, \sin^2 s\sin t, \sin s\cos s\rangle\,ds\,dt$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\sin^2 s\cos s\cos t + \sin^3 s\sin^2 t + \sin^2 s\cos s\cos t\right) ds\,dt$$

$$= \int_0^{2\pi} \left[2\cos t\,\frac{\sin^3 s}{3}\right]_0^{\pi} + \left[\sin^2 t\left(\frac{4}{3}\right)\right] dt$$

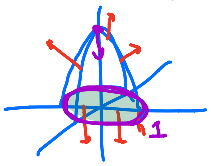
$$= \int_0^{2\pi} \frac{4}{3}\sin^2 t\,dt = \frac{4\pi}{3}$$

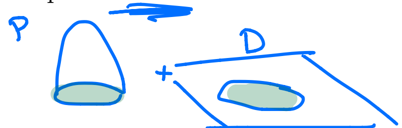
Remark 7.10.

Unless otherwise specified assume that your closed surfaces are always positively oriented (outward).

Example 7.11.

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$





$$\iint_P F \cdot d\vec{P} = \iint_D \langle y, x, z \rangle \cdot \pm \langle -2x, -2y, -1 \rangle \, dy\,dx$$
$$= \iint_D \langle y, x, 1-x^2-y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dy\,dx$$
$$= \iint_D 2xy + 2xy + 1 - x^2 - y^2 \, dy\,dx$$
$$= \int_0^{2\pi}\int_0^1 \left[4r^2 \sin\theta\cos\theta + 1 - r^2\right] r \, dr\,d\theta$$
$$= \int_0^{2\pi} \left[\sin\theta\cos\theta + \frac{1}{2} - \frac{1}{4}\right] d\theta$$
$$= \left[\frac{\sin^2\theta}{2} + \frac{1}{4}\theta\right]_0^{2\pi}$$
$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

$$\iint_D F \cdot d\vec{D} = \iint_D \langle y, x, z \rangle \cdot \pm \langle 0, 0, -1 \rangle \, dy\,dx$$
$$\iint -z \, dy\,dx = \iint 0 \, dA = 0$$

$$\iint_S F \cdot d\vec{S} = \iint_P F \cdot d\vec{P} + \iint_D F \cdot d\vec{D} = \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$$

Remark 7.12. In the flux definition:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

We can interpret this calculation as summing up movement of particles induced by the vector field across the surface S (Hence the $\vec{F} \cdot \vec{n}$).

Group Work

1. Find the upward flux of $\mathbf{F} = \langle x, y, z \rangle$ across the portion of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 1$.

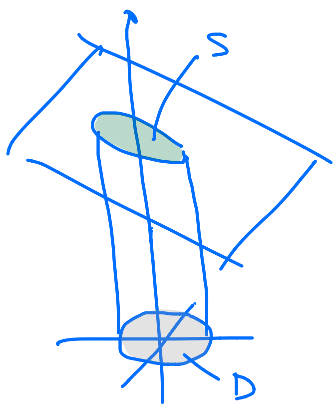
$z = 1 - x - y$

$$\iint_S \langle x, y, z \rangle \cdot d\vec{S} = \iint_D \langle x, y, 1-x-y \rangle \cdot \pm\langle -1, -1, -1 \rangle \, dA$$

$$= \iint_D \langle x, y, 1-x-y \rangle \cdot \langle 1, 1, 1 \rangle \, dA$$

$$= \iint_D (\cancel{x} + \cancel{y} + 1 - \cancel{x} - \cancel{y}) \, dA$$

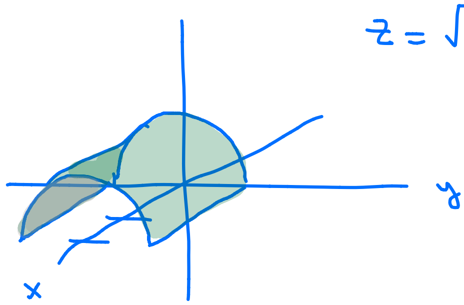
$$= \iint_D 1 \, dA = \boxed{\pi}$$



2. Evaluate the surface integral: $\displaystyle\iint_S \langle y^3, 4x, z^2\rangle \cdot d\mathbf{S}$

   where S is given by cylinder $z^2 + y^2 = 1$ above the xy-plane with positive orientation and $0 \le x \le 2$.

$z = \sqrt{1-y^2}$



$$\iint_S \langle y^3, 4x, z^2\rangle \cdot d\vec{S} = \underset{D}{} \langle y^3, 4x, z^2\rangle \cdot \pm\left\langle 0, \frac{-y}{\sqrt{1-y^2}}, -1\right\rangle dA$$

$$\iint_D \langle y^3, 4x, 1-y^2\rangle \cdot \left\langle 0, \frac{y}{\sqrt{1-y^2}}, 1\right\rangle dA$$

$$\iint_D \left[\frac{4xy}{\sqrt{1-y^2}} + 1 - y^2\right] dA$$

$$\int_0^2 \int_{-1}^{1} \left[\frac{4xy}{\sqrt{1-y^2}} + 1 - y^2\right] dy\, dx$$

$$= \int_0^2 \left[-4x\sqrt{1-y^2} + y - y^3/3\right]_{-1}^{1} dx$$

$$= \int_0^2 \left[1 - (-1) - \frac{1}{3} + \frac{(-1)}{3}\right] dx = \int_0^2 \frac{4}{3}\, dx$$

$$= \boxed{\frac{8}{3}}$$

8 Stokes' Theorem

Class Learning Goals

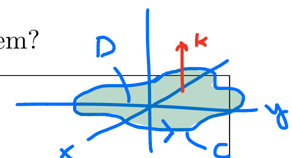
1. Understand the upgrade from Green's Theorem to Stokes' Theorem including the statement of Stokes' Theorem.
2. Practice turning surface integrals into line integrals
3. Practice turning line integrals into surface integrals

As we saw in 16.5 Green's Theorem can be expressed as

Green's Theorem C is a closed curve that bounds D in the xy plane.... blah blah other conditions.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \underbrace{(\text{curl }\mathbf{F})\cdot\mathbf{k}}_{Q_x-P_y}\, dA$$

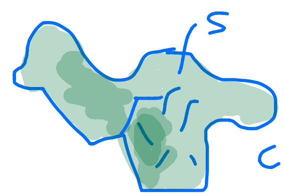
We can easily upgrade C to a curve in space bounding a surface S. $\mathbf{r}$ can parametrize C and $\mathbf{F}$ can be a vector field in three dimensions. The question becomes how do we upgrade K? What was special about it in Green's Theorem?



Theorem 8.1 (Stokes' Theorem)**.**

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\; dS$$



Remark 8.2.

It is important that $\mathbf{n}$ is an upward pointing unit normal vector and C is positively oriented when viewed from above. The can be generalized using the right-hand rule. As your fingers go around the curve your thumb will point in the direction of the unit normal vector.

Remark 8.3.

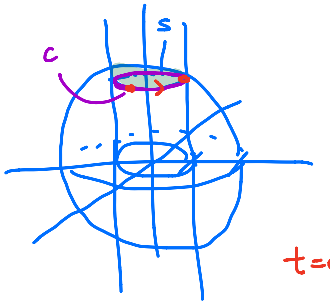
Unlike some theorems, Stokes' equal sign is really a two way street.

Example 8.4.

Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy-$plane.

$z^2 = 3$

$z = \sqrt{3}$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \langle xz, yz, xy \rangle \cdot d\vec{r}$$

$$r(t) = \langle \cos t, \sin t, \sqrt{3} \rangle \quad t \in [0, 2\pi]$$

$t = 0$

$$r'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$t = \frac{\pi}{2}$

$$\int_0^{2\pi} \langle xz, yz, xy \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$\int_0^{2\pi} \langle \cos t \sqrt{3}, \sin t \sqrt{3}, \sin t \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$\int_0^{2\pi} -\sqrt{3} \sin t \cos t + \sqrt{3} \sin t \cos t + 0 \, dt$$

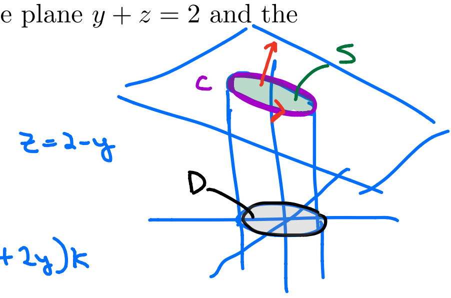
$$\int_0^{2\pi} 0 \, dt = 0$$

Remark 8.5.

This real difficulty in these problems identifying the boundary curve of the surface and making sure your parametrization orients the curve correctly.

Example 8.6.

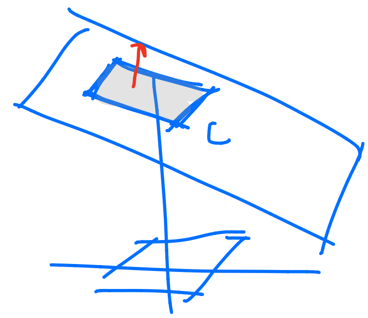
Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Note: C is to be oriented counterclockwise when viewed from above.)



$$\iint_S \underline{\text{curl}}\, F \cdot d\vec{S}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = (\ 0\ )i - (\ 0\ )j + (1+2y)k$$

$$= \langle 0,0,1+2y\rangle$$



$$\iint_D \langle 0,0,1+2y\rangle \cdot \pm\langle 0,-1,-1\rangle \, dA$$

$$\iint_D 1+2y \, dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\sin\theta\right] d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \pi$$

Remark 8.7.

If we wanted to evaluate the line integral in **Ex 8.6** we would end up integrating:
$\int_0^{2\pi} (\sin^3 t + \cos^2 t - 4\cos t + 4\sin t\cos t - \sin^2 t\cos t)dt.$

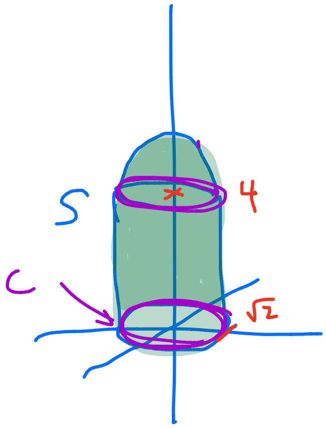
Remark 8.8.

The surface in **Ex 8.6** is not unique. However it is clearly the correct choice.

Group Work

Let S be the surface formed by capping the piece of the cylinder $x^2+y^2=2$, $0 \le z \le 4$ with the top half of the sphere $x^2+y^2+(z-4)^2=2$.

1. Draw a rough sketch of S.



2. What is $C = \partial S$? Parametrize C so that it has a positive orientation with respect to the outward normal.

$$C : r(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 0 \rangle$$
$$t \in [0, 2\pi]$$

3. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle \cancel{zx} + \cancel{z^2 y} + x, \cancel{z^3 yx} + y, z^4 x^2 \rangle$.

$$\| $$

$$\int_C F \cdot d\vec{r} = \int_0^{2\pi} \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 0 \rangle \cdot \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} 0 \, dt = \boxed{0}$$

8 Stokes' Theorem

Class Learning Goals

1. Tolerate the idea of the proof of Stokes' Thoerem.
2. Cement your knowledge of how to use Stokes' Thoerem.

Recall the statement of Stokes' Theorem

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

Today we will start by trying to gain an intuitive idea of what Stokes' Theorem is trying to convey.

Remark 8.9 (Idea of a Proof of Stokes' Theorem)**.**

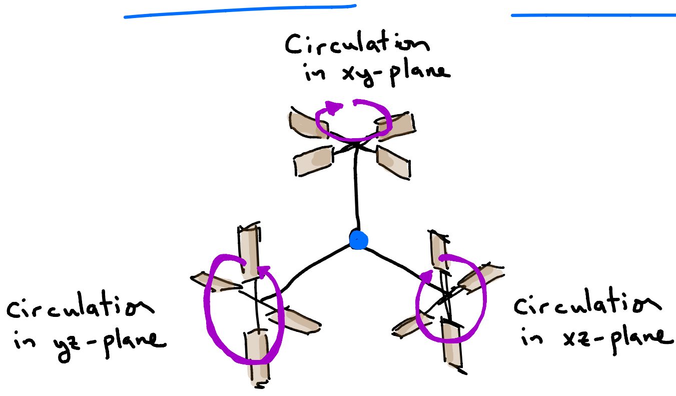
First we must verify that curl $\mathbf{F}$ has something to do with circulation.

Originally we considered circulation around a point when things rotated in the xy-plane perpendicular to $\mathbf{k}$.

Now there could be circulation in the xz-plane perpendicular to $\mathbf{j}$ and circulation in the yz-plane perpendicular to $\mathbf{i}$. We want to consider all three types of circulation. To help us we will create a vector to try to capture all these pieces of information.

$$(\text{Circulation in the } yz\text{-plane})\mathbf{i} + (\text{Circulation in the } xz\text{-plane})\mathbf{j} + (\text{Circulation in the } xy\text{-plane})\mathbf{k}$$

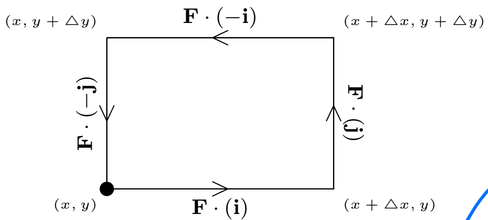
Picture:



We already know from before that:

$$(\text{Circulation in the } xy\text{-plane}) \;= Q_x - P_y$$

Spending a long time drawing pictures of rectangles like these:



We can get the circulation in the other two planes. Mainly:

$$(\text{Circlulation in the } xz\text{-plane}) \;= P_z - R_x$$

$$(\text{Circulation in the } yz\text{-plane}) \;= R_y - Q_z$$

And we can see there is some beautiful symmetry happening here.

So we get our mega vector that considers the circulation in all three coordinate planes:

$$\text{Mega Circulation Vector } = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

But **Mega Circulation Vector** isn't very official and won't make it into any math books so instead we recognize it as curl **F**.

It helps us measure the rate of rotation that is occurring at every point in the vector field.

To finish up we need to remember that Circulation (even at a point) needs to be a number

(Recall 0 circulation means no rotation, + circulation is counterclockwise, - is clockwise) So we need to turn this Mega Circulation Vector into a number...
In addition since our "water, beads, particles, etc" are trapped on a surface we really don't care about certain directions.

So it makes a certain amount of sense to have:

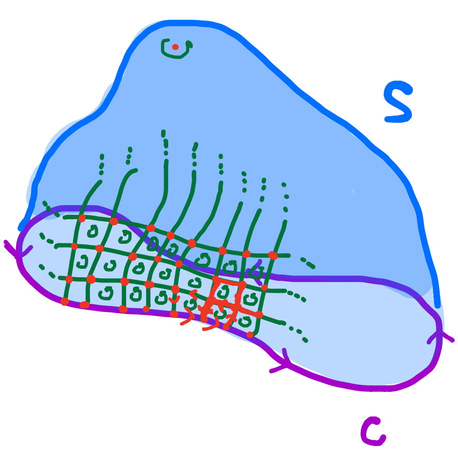
$$\text{Circulation at a point } = \text{curl } \mathbf{F} \cdot \mathbf{n}$$

where **n** is a unit normal vector to the surface's tangent plane at that point.

And this is how we finally see that curl $\mathbf{F} \cdot \mathbf{n} =$ Circulation at a pt.
Now we can use a similar argument to Green's Thoerem to get

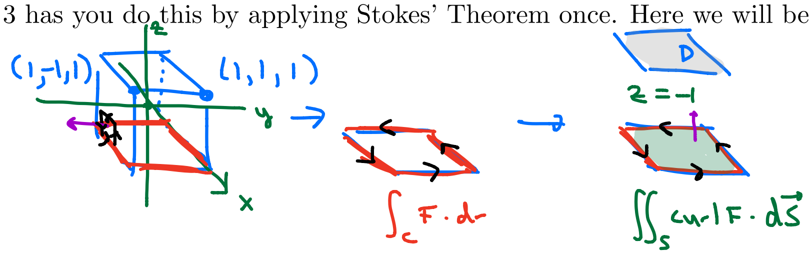
$$\sum(\text{Circulation around boundary of } S) = \sum(\text{Circulation around each point in } S)$$

$$\int_C \vec{F} \cdot d\vec{r} \quad = \iint_S \text{curl } \vec{F} \cdot \vec{n} \; dS$$



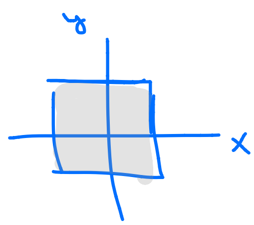
Example 8.10.

Evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ where S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.
Note: WW # 3 has you do this by applying Stokes' Theorem once. Here we will be extra clever and apply it twice!



$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & xy & x^2yz \end{vmatrix} = (x^2z - 0)i - (2xyz - xy)j + (y - xz)k$$
$$= \langle x^2z,\ xy - 2xyz,\ y - xz \rangle$$

$$\iint_S \langle x^2z, xy - 2xyz, y - xz \rangle \cdot d\vec{S} = \iint_D \langle -x^2, 3xy, x+y \rangle \cdot \pm\langle 0,0,-1 \rangle\, dA$$



$$= \iint_D x + y\, dA$$
$$= \int_{-1}^{1}\int_{-1}^{1} x + y\, dy\, dx$$

$$\int_{-1}^{1}\int_{-1}^{1} (x+y)\, dy\, dx = \int_{-1}^{1} \left[xy + \frac{y^2}{2}\right]_{-1}^{1} dx = \int_{-1}^{1} \left[x + \frac{1}{2}\right] - \left[-x + \frac{1}{2}\right] dx$$
$$= \int_{-1}^{1} 2x\, dx = \left[x^2\right]_{-1}^{1} = [1] - [1] = \boxed{0}$$

Example 8.11.

$\int_C \vec{F} \cdot d\vec{r}$ $\quad \vec{F} = \langle y, 0, 0 \rangle$

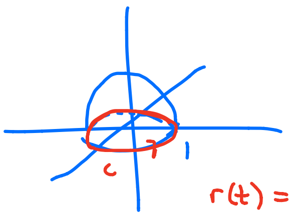
$\|$

12. (14 pts) on SS01 Final Exam.

Use Stokes' Theorem to evaluate $\iint_S \nabla \times (y\mathbf{i}) \cdot d\mathbf{S}$ where S is the hemisphere: $x^2 + y^2 + z^2 = 1, z \geq 0$.

Remember for quizzes & Final

$$\boxed{\cos^2 u = \frac{1+\cos(2u)}{2}, \qquad \sin^2 u = \frac{1-\cos(2u)}{2}}$$



$r(t) = \langle \cos t, \sin t, 0 \rangle$

$t \in [0, 2\pi]$

$$\int_C \langle y, 0, 0 \rangle \cdot dr$$

$$= \int_0^{2\pi} \langle \sin t, 0, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

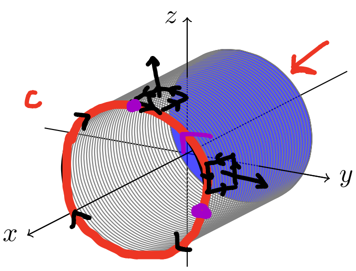
$$= \int_0^{2\pi} -\sin^2 t \, dt = -\int_0^{2\pi} \frac{1-\cos(2t)}{2} \, dt$$

$$= \vdots$$

$$= \boxed{-\pi}$$

Group Work

1. (15pts on E4 - FS14) Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = xyz\mathbf{i} + 2xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the cylinder $y^2 + z^2 = 1$, $x \in [-1,1]$ along with the disk $y^2+z^2 \le 1, x = -1$, oriented outward, shown to the right.



(a) Identify and parametrize the boundary curve of S with the correct orientation.

$r(t) = \langle 1, \cos t, \sin t \rangle$

$t \in [0, 2\pi]$

$t=0 \quad r(0) = \langle 1, 1, 0 \rangle$

$t = \frac{\pi}{2} \quad r(\frac{\pi}{2}) = \langle 1, 0, 1 \rangle$

(b) Write $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ as an equivalent line integral and then evaluate.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle xyz, 2xy, x^2yz \rangle \cdot \langle 0, -\sin t, -\cos t \rangle \, dt$$

$$= \int_0^{2\pi} \langle -\sin t \cos t, 2\cos t, -\sin t \cos t \rangle \cdot \langle 0, -\sin t, -\cos t \rangle \, dt$$

$$= \int_0^{2\pi} -2\sin t \cos t + \sin t \cos^2 t \, dt$$

$$= \left[-\sin^2 t - \frac{\cos^3 t}{3} \right]_0^{2\pi} = \boxed{0}$$

9 The Divergence Theorem

Class Learning Goals

1. Understand the statement of the Divergence Theorem and when it can be applied.
2. Apply the Divergence Theorem to problems

Let's quickly upgrade the alternate version of Green's Theorem so we can start doing some problems.

Recall from 16.5

Green's Theorem Alternate Bunch of conditions up here.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$
$$= \iint_D \operatorname{div} \mathbf{F}\, dA$$

This can be naturally upgraded to

Theorem 9.1 (The Divergence Theorem)**.**

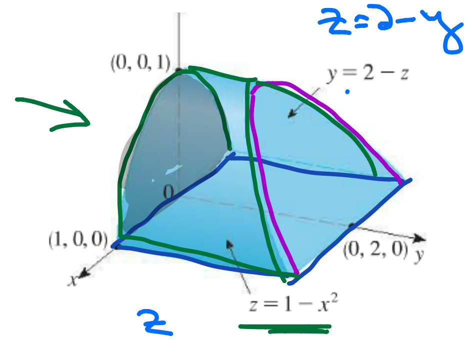
Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

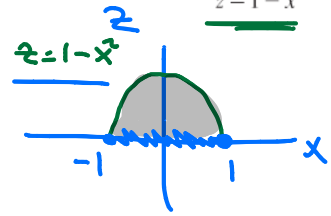
$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \operatorname{div} F\, dV$$

The main condition here is that S needs to be closed.

Example 9.2.

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.





$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint y + 2y + 0 \, dV$$

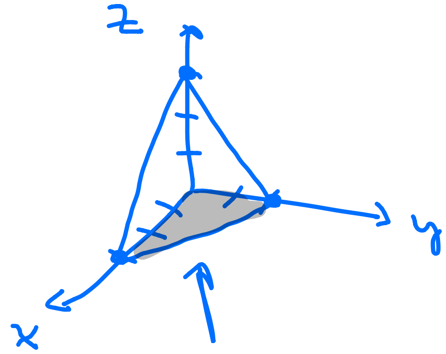
$$= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} 3y \, dy \, dz \, dx$$

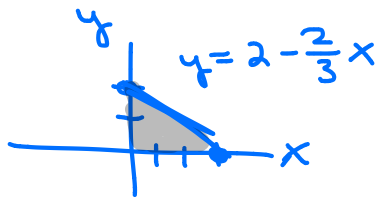
$$= \vdots$$

$$= 184/35$$

Example 9.3 (FS 01 Final Exam)**.**

12. (16 pts) E is a solid region in the first octant that lies beneath the plane $2x+3y+2z=6$. Let S be the boundary of E (S consists of 4 triangles). If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ use the Divergence Theorem to write $\iint_S \mathbf{F}\cdot d\mathbf{S}$ as a triple integral. **Do not evaluate the integral**.





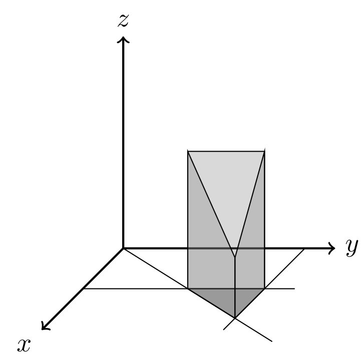
$$= \iiint_E \operatorname{div} F \, dV$$

$$\iiint_E 2x+2y+2z \, dV$$

$$\int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{(6-2x-3y)/2} (2x+2y+2z) \, dz \, dy \, dx$$

Example 9.4 (SS14 Exam 4 Question)**.**

(18 points) Consider the surfaces S from Exam 3 shown below:



$$x = \sqrt{3} \qquad y = 3$$
$$y = x \qquad z = 4-x$$
$$z = 0$$

Calculate the flux of $\mathbf{F} = (3x + \tan y)\mathbf{i} + (y - \ln(z+1))\mathbf{j} + (3xy - 2z)\mathbf{k}$ outward through S. (**Hint:** the volume enclosed by S is $24 - 13\sqrt{3}$)

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n} \, dS = \iiint_E \operatorname{div} F \, dV$$
$$= \iiint_E 3+1-2 \, dV$$
$$= \iiint 2 \, dV = 2 \iiint_E 1 \, dV$$
$$= \boxed{2\left[24-13\sqrt{3}\right]}$$

Group Work

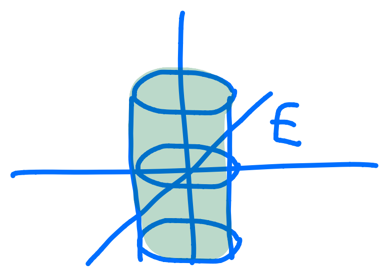
Consider $\mathbf{F} = \left\langle \frac{xy^2}{2}, \frac{y^3}{6}, zx^2 \right\rangle$ over the surface S, where S is the cylinder $x^2 + y^2 = 1$ capped by the planes $z = \pm 1$.

1. Is the net flux of $\mathbf{F}$ from the surface positive or negative?

$$\text{div } F = \frac{y^2}{2} + \frac{y^2}{2} + x^2 = \underline{x^2 + y^2}$$

since $x^2+y^2 \geq 0$ always
the net Flux is $(+)$.

2. What is the value of the flux across S?



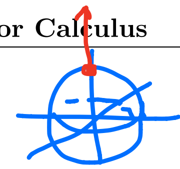
$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \text{div } F\, dV$$

$$= \iiint_E x^2+y^2\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\, r\,dz\,dr\,d\theta$$

$$= 2\pi \left[\frac{r^4}{4}\right]_0^1 \left[z\right]_{-1}^1$$

$$= 2\pi\left[\frac{1}{4}\right]\left[2\right] = \boxed{\pi}$$



Challenging problem

3. Evaluate $\iint_{S=\partial R} (x + y^2 + 2z)\, dS$, where R is the solid sphere $x^2 + y^2 + z^2 \le 4$ using the divergence theorem.

$$= \iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div} F\, dV$$

$F \cdot n = x + y^2 + 2z$

n is a unit normal vector for $S: x^2+y^2+z^2=4$ pointing outward

$\langle 2x, 2y, 2z\rangle$ is a normal vector that is outward pointing

$$\frac{\langle 2x, 2y, 2z\rangle}{\sqrt{4x^2+4y^2+4z^2}} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}} = \left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\rangle$$

So $\langle P, Q, R\rangle \cdot \left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\rangle = x + y^2 + 2z$

$P = 2 \quad Q = 2y \quad R = 4$

$$\iiint_E 2\, dV = 2\left[\tfrac{4}{3}\pi 2^3\right] = \boxed{\frac{64\pi}{3}}$$

9 The Divergence Theorem

Class Learning Goals

1. Pay our respects by going through a proof of the Divergence Theorem
2. Try a few more Divergence Theorem Problems

The Divergence Theorem

Let E be a solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_E \text{div}\,\mathbf{F}\, dV$$

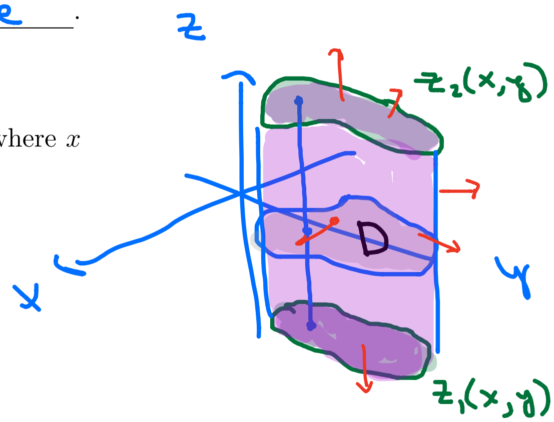
Idea of Proof Here I will give a more rigorous proof then I do normally. Those pursuing a degree in Mathematics should pay extra attention to this proof technique as it is a common technique used again and again, that is:

1. Proof in a special case
2. How to expand the special case to a general region
3. How to expand the special case to a general vector field

Proof in special case: $\mathbf{F} = \langle 0, 0, R\rangle$ and E is vertically simple.

E is called vertically simple if:

1. E is bounded on top by $z = z_2(x, y)$ and on bottom by $z = z_1(x, y)$ where x and y for these surfaces are over the same planar region D.
2. E includes all line segments from $z_1(x_0, y_0)$ to $z_2(x_0, y_0)$ where x_0, y_0 are in D.



Now let's expand the right hand side of the divergence theorem

$$\begin{aligned}\iiint_E \text{div}\,\mathbf{F}\, dV &= \iiint R_z(x,y,z)\, dV \\ &= \iint\int_{z_1(x,y)}^{z_2(x,y)} R_z(x,y,z)\, dz\, dy\, dx \\ &= \iint_D \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right] dy\, dx\end{aligned}$$

Now we evaluate the left side to hopefully get the same thing:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{Bottom}} F\cdot n\,dS + \iint_{\text{Sides}} F\cdot n\,dS + \iint_{\text{Top}} F\cdot n\,dS$$

$$= \iint_{\text{Bottom}} \langle 0,0,R\rangle \cdot \cancel{\pm}\left\langle \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}, -1\right\rangle dy\,dx$$

$$+ \iint_{\text{sides}} \langle 0,0,R\rangle \cdot \pm\langle \sim, \sim, 0\rangle\, dA$$

$$+ \iint_{\text{top}} \langle 0,0,R\rangle \cdot \left\langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, +1\right\rangle dy\,dx$$

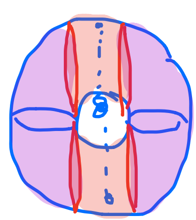
$$= \iint_{\text{Bottom}} -R(x,y,z)\,dy\,dx + \iint_{\text{sides}} 0\,dA + \iint_{\text{top}} R(x,y,z)\,dy\,dx$$

$$= \iint_D -R(x,y,z_1(x,y))\,dy\,dx + \iint_D R(x,y,z_2(x,y))\,dy\,dx$$

$$= \iint_D \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right] dy\,dx$$

How to expand the special case to a general region
Any region can be decomposed into into the sum of vertically simple regions.

Compute the surface integrals and triple integrals over each one.



How to expand the special case to a general vector field

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \text{div }\mathbf{F}\,dV$$

$$\iint_S \langle P,Q,R\rangle \cdot \langle n_1,n_2,n_3\rangle\,dS = \iiint_E (P_x+Q_y+R_z)\,dV$$

$$\iint_S Pn_1 + Qn_2 + Rn_3\,dS = \iiint_E (P_x+Q_y+R_z)\,dV$$

$$\iint_S Pn_1\,dS + \iint_S Qn_2\,dS + \underline{\iint_S Rn_3\,dS} = \iiint_E P_x\,dV + \iiint_E Q_y\,dV + \underline{\iiint_E R_z\,dV}$$

We have already shown that $\iint_S Rn_3 \, dS = \iiint_E R_z \, dV$. Similar proofs can be used to show

$\iint_S P n_1 \, dS = \iiint_E P_x \, dV$ and $\iint_S Q n_2 \, dS = \iiint_E Q_y \, dV$.

Example 9.5.

Evaluate the surface integral $\iint_S \langle xz, -2y, 3x \rangle \cdot d\mathbf{S}$ where S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation.

$$\iiint_E z - 2 + 0 \, dV = \iiint_E z \, dV - 2\iiint_E 1 \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^2 (\rho\cos\phi - 2)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 0 - 2\left[\tfrac{4}{3}\pi \, 2^3\right] = \boxed{-\frac{64\pi}{3}}$$

$$= 2\pi \int_0^{\pi}\int_0^2 \rho^3 \sin\phi\cos\phi - 2\rho^2 \sin\phi \, d\rho \, d\phi$$

$$= 2\pi \int_0^{\pi} \left[\frac{\rho^4}{4}\sin\phi\cos\phi - \frac{2}{3}\rho^3 \sin\phi\right]_0^2 d\phi$$

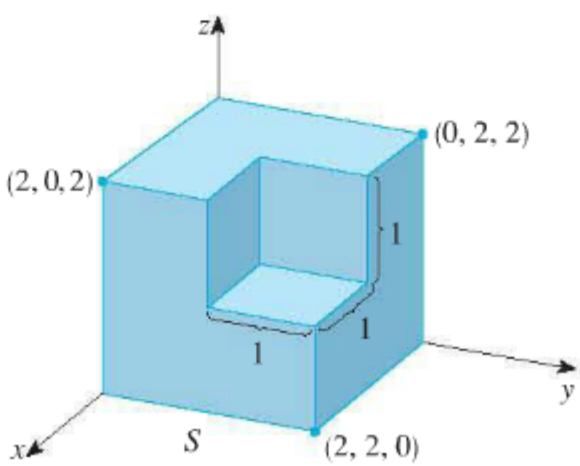
$$= 2\pi \int_0^{\pi} \left[4\sin\phi\cos\phi - \frac{16}{3}\sin\phi\right] d\phi$$

$$= 2\pi \left[2\sin^2\phi + \frac{16}{3}\cos\phi\right]_0^{\pi}$$

$$= 2\pi\left[\frac{16}{3}(-1) - \frac{16}{3}(1)\right] = 2\pi\left[\frac{16}{3}\right][-2] = \boxed{-\frac{64\pi}{3}}$$

Example 9.6.

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the outwardly oriented surface shown in the figure below.



$$\iiint_E 3 \, dV$$

$$3\iiint_E 1 \, dV$$

$$3[8 - 1] = \boxed{21}$$

Example 9.7.

Prove that $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = 0$ assuming S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

$$\| $$

$$\iiint_E \text{div}(\text{curl}\, F)\, dV = \iiint_E 0\, dV$$

$$= 0$$

Example 9.8.

Use the Divergence Theorem to evaluate $\iint_S (2x + 2y + z^2)\, dS$ where S is the sphere $x^2 + y^2 + z^2 = 1$.

$$n = \langle 2x, 2y, 2z \rangle$$

$$n = \langle x, y, z \rangle \leftarrow$$

$$\iint \langle \overset{2}{P}, \overset{2}{Q}, \overset{z}{R} \rangle \cdot \langle x, y, z \rangle\, dS = \iint_S F \cdot n\, dS$$

$$= \iiint_E \text{div}\, F\, dV$$

$$= \iiint_E 1\, dV$$

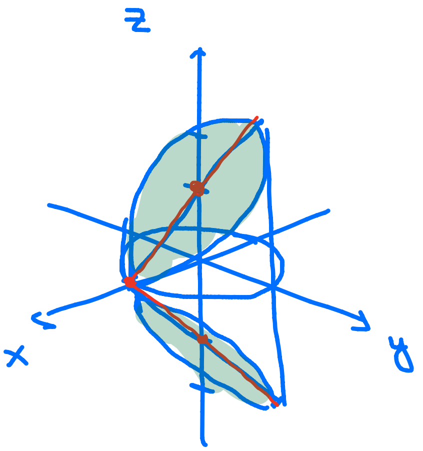
$$= \boxed{\frac{4}{3}\pi}$$

Group Work

1. Compute $\iint_{S=\partial R} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = (x-z)\mathbf{i} + (y-x)\mathbf{j} + (z-y)\mathbf{k}$

   and S is the cylinder $x^2+y^2=1$ capped by the planes $2z = 1-x$ and $2z = x-1$.

$$\text{div } F = 1+1+1 = 3$$



$$\iiint_E 3 \, dV$$

$$3\int_0^{2\pi}\int_0^1\int_{\frac{r\cos\theta-1}{2}}^{\frac{1-r\cos\theta}{2}} r\,dz\,dr\,d\theta$$

$$3\int_0^{2\pi}\int_0^1 r\left[\frac{1-r\cos\theta}{2} - \left[\frac{r\cos\theta-1}{2}\right]\right] dr\,d$$

$$3\int_0^{2\pi}\int_0^1 r\left[1-r\cos\theta\right] dr\,d\theta$$

$$3\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\cos\theta\right]_0^1 d\theta$$

$$3\int_0^{2\pi}\left[\frac{1}{2} - \frac{1}{3}\cos\theta\right] d\theta$$

$$3\left[\frac{1}{2}\right]\left[2\pi\right] = 3\pi$$