Inner Product Space Results

Lemma A. Once an orthonormal basis $u_1, u_2, \ldots, u_n$ is chosen for V, the inner product $\langle \cdot, \cdot \rangle$ on V is given merely as ordinary dot product of the coordinates. That is, if

$$v = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$$

and

$$w = y_1 u_1 + y_2 u_2 + \cdots + y_n u_n,$$

then

$$\langle v, w \rangle \; = \; x^T y \; = \; x_1 y_1 + \cdots + x_n y_n. \tag{1}$$

Proof. Because of orthonormality,

$$\langle v, w \rangle \; = \; \langle \sum_i x_i u_i, \sum_j y_j u_j \rangle = \sum_{i,j} x_i \langle u_i, u_j \rangle y_j = \sum_k x_k y_k = x^T y.$$

Definition. The operator T^* defined by the relation

$$\langle Tu, v \rangle = \langle u, T^* v \rangle \tag{2}$$

for all u, v is called the operator *adjoint* to T.

Lemma B. For each linear operator T, the relation (2) defines a unique linear operator T^*.

Proof. We assume that the inner product space is *self-dual,* i.e., that each bounded linear functional $f : V \longrightarrow \mathbf{R}$ is of the form $f(v) = \langle v, w \rangle$ for some w in V. (This certainly holds when V has an orthonormal basis (exercise).)

For each fixed v, the map $f(u) = \langle Tu, v \rangle$ is a linear functional into $\mathbf{R}$ and hence is by self-duality of the form $f(u) = \langle u, w \rangle$ for some (unique) w. Define $Tv = w$. Now check that T^* is linear.

Lemma C. For a real inner product space, when an operator T is represented by the matrix A with respect to some orthonormal basis, then the matrix representing its adjoint T^* with respect to this same basis is A^T.

Proof. By Lemma A, $\langle Ax, y\rangle = (Ax)^T y = x^T A^T y = \langle x, A^T y\rangle$. But also, $\langle Ax, y\rangle = \langle x, A^* y\rangle$, giving $A^T y = A^* y$ for all y.

Remark. The expected similar result holds for complex inner product spaces: The adjoint is represented by the conjugate tranpose.

Lemma D. The spectrum of a self-adjoint operator is real.

Proof. Let T be self-adjoint and represented by the (symmetric) matrix A with respect to some orthonormal basis. We may identify the (real) inner product space V with $\mathbf{R}^n$ under the natural basis. Lift this matrix A to a matrix map of $\mathbf{C}^n$. Let λ be a (possibly) complex eigenvalue of A belonging to the (possibly) complex vector v. Then

$$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle = \langle v, A^* v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \langle v, v\rangle\overline{\lambda}.$$

Thus λ is real.

The Spectral Theorem for $\mathbf{R}^n$. A self-adjoint operator possesses an orthonormal basis of eigenvectors. That is, every symmetric matrix A can be brought orthogonally to diagonal form

$$Q^{-1}AQ \;=\; diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

where $Q^{-1} = Q^T$.

Proof. By the fundamental theorem of algebra and lemma D, the characteristic equation of A factors completely into linear factors over $\mathbf{R}$; all of its eigenvalues are real. Let $Au_1 = \lambda_1 u_1$, where $\|u_1\| = 1$ Let $V_1 = span\{u_1\}^\perp$. Note that V_1 is invariant under A since if $v_1 \perp u_1$, then $\langle Av_1, u_1\rangle = \langle v_1, Au_1\rangle = \langle v_1, \lambda_1 u_1\rangle = \lambda_1\langle v_1, u_1\rangle = 0$. Find an orthonormal basis $v_2, \ldots, v_n$ for V_1.

With respect to the orthonormal basis $u_1, v_2, \ldots, v_n$, A now has matrix

$$Q^{-1}AQ \;=\; \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix},$$

where A_1 is self-adjoint. Proceed in a similar manner with A_1.

Definition. The *numerical range* of an operator A is the set of all values $\langle Au, u\rangle$ taken on as u ranges over the unit sphere $\|u\| = 1$. The *numerical radius* $\gamma(A)$ is the radius of the smallest circle containing the numerical range.

Corollary A. For any operator A, $\rho(A) \le \gamma(A) \le \|A\|_2$.

Proof. If $Au = \lambda u$, $\|u\| = 1$, then $\langle Au, u\rangle = \lambda$, hence the spectrum of A is among the numerical range of A. Hence $\rho(A) \le \gamma(A)$. The remaining inequality follows from the Cauchy inequality:

$$|\langle Au, u\rangle| \ \le \|Au\|\|u\| \ = \ \|Au\| \ \le \ \|A\|.$$

Corollary B. The spectral radius of a self-adjoint operator A coincides with its 2-norm, i.e.,

$$\rho(A) \ = \ \gamma(A) \ = \ \|A\|.$$

Proof. For any operator A and any induced norm, we have $\rho(A) \ \le \ \|A\|$.

Conversely, since orthogonal changes of bases preserve 2-norms of vectors, they also preserve the induced 2-norms of operators. Thus

$$\|Q^{-1}AQ\| \ = \ \|A\|,$$

so we may as well assume via the spectral theorem that A is already diagonal:

$$A \ = \ diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

Renumber so that $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$. Thus $\rho(A) = |\lambda_1|$.

Then for $\|x\| = 1$,

$$\|Ax\|^2 \ = \ \|(\lambda_1\, x_1, \lambda_2\, x_2, \ldots, \lambda_n\, x_n)^T\|^2$$

$$= \ \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \cdots + \lambda_n^2 x_n^2 \ \le \ |\lambda_1|^2 x_1^2 + |\lambda_1|^2 x_2^2 + \cdots + |\lambda_1|^2 x_n^2$$

$$= \ |\lambda_1|^2 (x_1^2 + \cdots + x_n^2) = |\lambda_1|^2 = \rho(A)^2.$$

Thus $\|A\| = \sup_{\|x\|=1} \|Ax\| \le \rho(A)$.

Result. For any operator A, its 2-norm is given by the formula

$$\|A\| \;=\; \sqrt{\rho(A^T A)}.$$

Proof. By corollary B, the spectral radius, numerical radius, and norm of the symmetric $A^T A$ all coincide. But then

$$\|A\|^2 = \sup_{\|u\|=1} \langle Au, Au\rangle = \sup_{\|u\|=1} \langle u, A^T Au\rangle = \gamma(A^T A) = \rho(A^T A).$$

Result. Eigenvectors belonging to distinct eigenvalues of a self-adjoint operator are orthogonal.

Proof. Suppose $Au = \lambda u$ and that $Av = \mu v$. Then

$$\begin{aligned} \langle Au, v\rangle &= \langle \lambda u, v\rangle = \lambda\langle u, v\rangle \\ &= \langle u, Av\rangle = \langle u, \mu v\rangle = \mu\langle u, v\rangle. \end{aligned}$$

Thus either $\langle u, v\rangle = 0$ or $\lambda = \mu$.