**The Spectral Mapping Theorem.** Let $A$ be an operator on an $n$-dimensional real or complex vector space $V$ and let $q(x)$ be any polynomial. Then the spectrum of the polynomial operator $q(A)$ is the image of the spectrum of $A$ under $q$, i.e.,

$$sp(q(A)) = q(sp(A)).$$

As an epigram, *where goes the operator, so goes its spectrum.*

Moreover, multiplicities are preserved—the multiplicity of the eigenvalue $\mu$ of $q(A)$ is the sum of the multiplicities of the eigenvalues $\lambda$ of $A$ mapped by $q$ to $\mu$.

**Proof.** Since under any change of basis $P$,

$$P^{-1}q(AP) = q(P^{-1}AP),$$

we may as well assume that $A$ is already in Jordan canonical form. But then $q(A)$ consists of diagonal blocks of lower-triangular matrices with diagonal elements $q(\lambda)$, where $\lambda$ are the eigenvalues of $A$. But the eigenvalues of $q(A)$ are these diagonal entries $q(\lambda)$. Note that multiplicities are completely determined: $\mu = q(\lambda)$ will occur along the diagonal of $q(A)$ the exact number of times that $\lambda$’s with $q(\lambda) = \mu$ occur along the diagonal of $A$.

**Remark A.** The spectral mapping theorem holds for any finite dimensional vector space $V$ over any field $K$ since we may embed $K$ into the splitting field $\Sigma$ of the characteristic polynomial of $A$, lift $V$ to a vector space with scalars in $\Sigma$, so that the Jordan canonical form obtains.

**Remark B.** These polynomial mappings, however, may destroy invariant subspace structure. For example, a (cyclic) $4 \times 4$ operator $A$ with minimal polynomial $m(x) = \phi(x) = x^4$, when squared $q(A) = A^2$, has invariant subspace structure $x^2, x^2$, as can be seen by squaring the Jordan canonical form for $A$ and calculating elementary divisors.