Principal Moments of Inertia

The *moment of inertia* $I_{\mathbf{u}}$ of a solid body V rotating about an axis through the origin in the direction given by the unit vector $\mathbf{u}$ (counter clockwise when viewed point-on) is the scalar

$$I_{\mathbf{u}} = \int_V r^2\, dm = \int_V (\mathbf{x} \times \mathbf{u})^2\, dm = \int_V (\mathbf{x} \times \mathbf{u})^2\, \rho\, d\mathbf{x}, \tag{1}$$

where $\rho = \rho(\mathbf{x})$ is mass density. The rotational kinetic energy of this body is then

$$T = \frac{\omega^2 I_{\mathbf{u}}}{2}, \tag{2}$$

where ω is the angular velocity of rotation.

By carrying out the integrations in (1) this moment of inertia is realized by a quadratic form:

$$I_{\mathbf{u}} = \mathbf{u}^T A \mathbf{u}, \tag{3}$$

where $A^T = A$. By the spectral theorem we may perform a distance-preserving change of coordinates $P = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, $P^T P = I$, so that the form (3) giving the moment of inertia (1) is now diagonal:

$$I_{\mathbf{u}'} = \mathbf{u}'^T \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \mathbf{u}' \tag{4}$$

with $I_1 \geq I_2 \geq I_3 > 0$. The eigenvalues I_1, I_2, I_3 are called the *principal moments of inertia.* Their associated eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are called the *principal axes.*[1]

Example. Think about an American football: One principal axis $\mathbf{u_3}$ is along the long axis, and the other two principal axes $\mathbf{u}_1, \mathbf{u}_2$ are in the 'equatorial plane,' with $I_1 = I_2 > I_3$.

The Charon Principal. If $I_1 > I_2 > I_3$, then rotation about the intermediate axis is unstable—asteroids tumble! (See Ian Stewart.)

Question. Are we sure that the orthogonal change of coordinates P has preserved the right-handedness of the coordinate system?

[1] This is historically the form in which the spectral theorem first appeared.

Fact. Suppose we have brought the symmetric 3×3 matrix A to diagonal form:

$$P^{-1}AP \;=\; \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \;=\; \Lambda,$$

where P is orthogonal, i.e., $P^{-1} = P^T$. Then there is a more careful choice of P that is a rotation about some axis.

Proof. We may exchange two columns of P, if necessary, to ensure that P has determinant 1. Swapping two columns of P only changes the order of two of the eigenvalues λ_i along the diagonal of Λ. In symbols, if E is the elementary matrix swapping the two columns, then because $AP = P\Lambda$, we have $APE = P\Lambda E = PE(E^T\Lambda E)$.

Because P is orthogonal, all its eigenvalues lie on the unit circle. Since $n = 3$, one or all three must be real. But since P has determinant 1, one eigenvalue belonging to a unit eigenvector w_0 must be 1. We now show P is rotation about w_0.

Because P is orthogonal, it is normal, and hence eigenspaces of P are reducing. In particular, the subspace S orthogonal to w_0 is invariant under P. Find two perpendicular unit vectors $u_0, v_0 \in S$. Then with respect this basis P has matrix

$$O^{-1}PO \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},$$

where the lower right 2×2 block Q has determinant 1.

The othogonal map P when cut back to S is again orthogonal, hence

$$Q^{-1} \;=\; \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \;=\; Q^T \;=\; \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Hence $a = d$ and $b = -c$ giving $1 = ad - bc = a^2 + b^2$. Thus $a = \cos\theta$ and $b = -\sin\theta$. Thus P cut back to the plane S is a rotation. Hence P is a rotation about w_0.