Divide into groups of two or three. Solve as many of the following problems as time allows. Remember that its better to solve fewer questions thoroughly than solve more with less precision. The ones that you dont get to can be used to help you prepare for exams. After writing each proof, read it through and discuss:

1. Does the proof contain all of the essential elements of the proof type used?
2. Is the work organized in a clear and concise manner that allows the reader to easily follow the logic?
3. Are there any sentences or phrases in the proof that are irrelevant or unnecessary?
4. Are any variables overloaded? In other words, were any variable names used twice for two different quantities, when they should have been given different names? Make improvements to your work based on this.

Definition 1.1. For two sets X and Y

- they have the same cardinality ($|X| = |Y|$) if there exists a bijection $f : X \to Y$.
- the cardinality of X is less that or equal to the cardinality of Y ($|X| \leq |Y|$) if there is an injection $g : X \to Y$.

Definition 1.2. A set X is

- countable if $|X| \leq |\mathbb{N}|$
- countably infinite if $|X| = |\mathbb{N}|$

Theorem 1.3. An infinite set is countable if and only if all of its elements can be written in an infinite list.

Theorem 1.4. Let X be a countable set. Then either

- $|X| = m$ for some $m \in \mathbb{N}$, or
- X is countably infinite.

You will also need the definition of power set:

Definition 1.5. Let X be a set. Then we define the *power set* of X, $\mathcal{P}(X)$, to be the set of all subsets of X

$$\mathcal{P}(X) := \{Y \mid Y \subseteq X\}$$

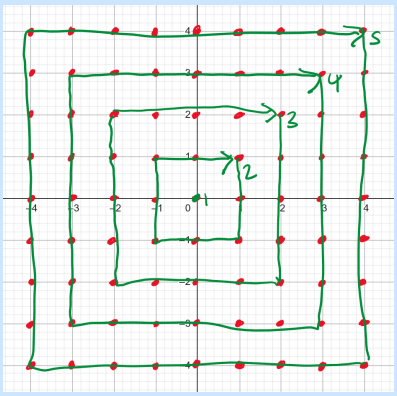
For example

$$\mathcal{P}(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$$
$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Question 1. Show that $\mathbb{Z} \times \mathbb{Z}$ is countable.

Hint: the following illustration of $\mathbb{Z} \times \mathbb{Z}$ may be helpful, though your proof needs to be more than drawing on the diagram.

Solution:
We show that we can write the elements of $\mathbb{Z} \times \mathbb{Z}$ in an infinite list, in the order illustrated on the diagram.



For each $k \in \mathbb{N} \cup \{0\}$, we make a finite list consisting of exactly those elements of $\mathbb{Z} \times \mathbb{Z}$ which have maximal absolute value of their entries equal to k, by following the arrows clockwise around the square from the top right corner:

$$(k,k),(k,k-1),\dots(k,-(k-1)),(k,-k)$$

$$(k-1,-k),(k-2,-k),\dots,(-(k-1),-k),(-k,-k)$$

$$(-k,-(k-1)),(-k,-(k-2)),\dots,(-k,k-1),(-k,k)$$

$$(-(k-1),k),(-(k-2),k),\dots(k-1,k)$$

We now list all element of $\mathbb{Z} \times \mathbb{Z}$ by appending these lists in increasing order of k. In this way we produce an infinite list consisting of every element of $\mathbb{Z} \times \mathbb{Z}$ exactly once. Hence $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Question 2. An integer sequence is an infinite ordered list $(a_1, a_2, a_3, \dots)$ whose entries consist of integers. Two integer sequences $(a_1, a_2, a_3, \dots)$ and $(b_1, b_2, b_3 \dots)$ are equal if $a_i = b_i$ for all i.

Show that the set S of integer sequences is uncountable.
Hint: use Cantor's diagonal argument

Solution: Suppose S is countable. Then by Theorem 1.3 we can write the elements of S as an infinite list:

$$\begin{aligned} A_1 &= (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, \dots) \\ A_2 &= (a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}, \dots) \\ A_3 &= (a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, \dots) \\ A_4 &= (a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}, \dots) \\ &\vdots \end{aligned}$$

where each $a_{i,j} \in \mathbb{Z}$. For contradiction, we aim to produce an integer sequence $B = (b_1, b_2, b_3, b_4, \dots)$ which is not on the list.
Define B by

$$b_n = \begin{cases} 0 & \text{if } a_{n,n} = 1 \\ 1 & \text{if } a_{n,n} \neq 1 \end{cases}$$

B is not on the list, because it differs from A_n in the n^{th} element for each $n \in \mathbb{N}$. Hence the list could not have contained every element of S, and so S is uncountable.

Question 3. Show that the union of two countable sets A and B is countable.
Hint: Alternate between lists.

Solution: This is true if A and B are finite since their union is also finite. On the previous worksheet, we saw that this is true if one of the sets is finite and other countably infinite. So assume that both are countably infinite.

Because A is countably infinite, we can list its elements as a sequence $a_1, a_2, a_3, \ldots$. Because B is countably infinite, we can list its elements as a sequence $b_1, b_2, b_3, \ldots$. Then the alternating sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$$

contains all elements of $A \cup B$. This implies that $A \cup B$ is countable by Theorem 1.3.

Question 4. Show that the set Y of all *finite* subsets of $\mathbb{N}$ is countably infinite.

$$|\{Z \in \mathcal{P}(\mathbb{N}) \mid |Z| < \aleph_0\}| = \aleph_0$$

Hint: if Z is a finite subset of $\mathbb{N}$ then it is also contained in $\mathcal{P}(\{1, 2, \ldots, m\})$ where m is the maximal element of Z.

Solution: As mentioned in the hint, if Z is a finite subset of $\mathbb{N}$ then it is also a finite subset of $\mathcal{P}(\{1, ..., m\})$ for some m. Question 6 showed us that $\mathcal{P}(\{1, ..., m\})$ is finite. We can therefore write a list containing every finite subset of $\mathbb{N}$ by listing the finitely many elements if $\mathcal{P}(\{1\})$, then listing the finitely many elements in $\mathcal{P}(\{1, 2\})$, then listing the finitely many elements in $\mathcal{P}(\{1, 2, 3\})$ etc. More precisely we construct the infinite list by appending $\mathcal{P}(\{1, \ldots, k\})$ to the previous list for every $k \in \mathbb{N}$. Then 1.3 shows that Y is countable. It is not finite since there is an injection $f : \mathbb{N} \to Y$ given by $f(m) = \{m\}$. Hence $|Y| = \aleph_0$.

(Alternatively, we could have made a list with no repeats by instead repeatedly appending a list of all subsets whose largest element is k for each k. This would show directly that $|Y| = \aleph_0$.)

Question 5. Use the diagonal argument to show directly (i.e. without appealing to that result) that the set of all infinite subsets of $\mathbb{N}$ is uncountable. Hence deduce directly (i.e. without appealing to a more general result from class) that $\mathcal{P}(\mathbb{N})$ is uncountable.

Solution: Assume for contradiction that the set of infinite subsets of $\mathbb{N}$ is countable. Then we can write those subsets in a list: Furthermore, we can ensure that the elements of every set are listed in increasing order, since a set is not changed by reordering its elements.

$$A_1 = \{a_{1,1}, a_{1,2}, a_{1,3}, \dots\}$$
$$A_2 = \{a_{2,1}, a_{2,2}, a_{2,3}, \dots\}$$
$$A_3 = \{a_{3,1}, a_{3,2}, a_{3,3}, \dots\}$$

We now define a set

$$B = \{b_1, b_2, b_3, ...\}$$

by the iterative process:

$$b_1 = a_{1,1} + 1, \text{ and } b_i = \max(a_{i,i} + 1, b_{i-1} + 1) \text{ for } i > 1$$

Note that the definition ensures that the elements of the set B are also listed in increasing order. Therefore, B is not on the list because the i^{th} largest element in B (which is b_i by construction) is greater than the the i^{th} largest element in A_i for each i, which is $a_{i,i}$ and hence $B \neq A_i$. Therefore we reach a contradiction and so there are uncountably many infinite subsets of $\mathbb{N}$.

If $\mathcal{P}(\mathbb{N})$ were countable, we could write its elements in a list, but by deleting all finite subsets we would obtain a list of all the infinite subsets, contradicting the fact we just proved. Therefore $\mathcal{P}(\mathbb{N})$ is uncountable.

Question 6. Let X be a finite set. Show that $|\mathcal{P}(X)| = 2^{|X|}$. In particular, that it is always finite.

Solution: There are several ways to prove this statement. We will prove it by induction on $|X|$.

Base Case: Suppose $|X| = 0$. Then, $X = \emptyset$ and $\mathcal{P}(X) = \{\emptyset\}$. Hence, $|\mathcal{P}(X)| = 1 = 2^0$.

Inductive step: Suppose the statement is true for some $k \in \mathbb{N} \cup \{0\}$. That is, suppose that for all sets X with $|X| = k$, we have $|\mathcal{P}(X)| = 2^k$. Now suppose X is a set with $|X| = k + 1$. Let x_* be some element of X, and let $Y = X \setminus \{x_*\}$. Since $|Y| = k$, the inductive hypothesis implies that $|\mathcal{P}(Y)| = 2^k$. Now let

$$Y' = \{A \cup \{x_*\} \mid A \subseteq Y\}$$

Define $f : \mathcal{P}(Y) \to Y'$ by $f(A) = A \cup \{x_*\}$. Then, f is a bijection (we will leave this for you to check). Therefore, $|\mathcal{P}(Y)| = |Y'|$ and thus $|Y'| = 2^k$. Next, by the definition of power sets

$$\mathcal{P}(X) = \mathcal{P}(Y) \cup Y'$$

Also, since $x_* \notin Y$, we must have that $\mathcal{P}(Y) \cap Y' = \emptyset$. Therefore,

$$|\mathcal{P}(X)| = |\mathcal{P}(Y) \cup Y'| = |\mathcal{P}(Y)| + |Y'| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore, the statement follows by induction.

Recall that for any set X, we can find one of a strictly larger cardinality by taking the power set $\mathcal{P}(X)$. We can iterate this process to find an infinite sequence of strictly increasing cardinalities:

$$\aleph_0 = |\mathbb{N}| \lneq |\mathcal{P}(\mathbb{N})| \lneq |\mathcal{P}^2(\mathbb{N})| \lneq \dots$$

The aim of the next few problems is to find sets of absolutely enormous cardinality which are bigger than any of these.

Before we construct the promised huge sets, we need to define indexed unions:

Definition 1.6. Let I be a set. If we have a set X_i for every $i \in I$, we can define a set called the *indexed union* of the X_i given by

$$\bigcup_{i\in I} X_i = \{x \mid x \in X_i \text{ for some } i \in I\}.$$

I is called the *indexing set*.

Question 7. Suppose that we have sets X_i indexed by a set I. Show that for every $i \in I$ we have

$$|X_i| \leq |\bigcup_{i\in I} X_i| \tag{*}$$

and give examples for which I has at least two elements and:

- The X_i are disjoint and strict inequality holds in (*) for all $i \in I$.
- The X_i are disjoint and equality holds in (*) for some $i \in I$.
- $I = \mathbb{N}$, $X_i \subseteq X_{i+1}$ for all $i \in I$ and strict inequality holds in (*) for every $i \in I$.

Solution: First the inequality of cardinalities. We need to show that there is an injective function

$$f : X_i \to \bigcup_{i\in I} X_i$$

(because this is the defintion of $\leq$ for cardinalities).
But in this case, $X_i \subset \bigcup_{i\in I} X_i$ by definition of the indexed union. Hence we can just define $f(x) = x$ for all $x \in X_i$, where we view the latter x as living inside the copy of X_i inside the indexed union.

- $I = \{1, 2\}$, $X_1 = \{1\}$ and $X_2 = \{2\}$. The indexed union is just a regular union and $X_1 \cup X_2 = \{1, 2\}$ has cardinality 2, while both X_i have cardinality 1.
- $I = \{1, 2\}$, $X_1 = \{\text{all even numbers}\}$, $X_2 = \{\text{all odd numbers}\}$. Again since I has two elements the indexed union is just the regular union, and we have already seen that both X_1, X_2 and their union $\mathbb{N}$ have cardinality $\aleph_0$.
- Set $X_i = \{1, 2, \dots, i\}$. Then the indexed union is $\mathbb{N}$, which has larger cardinality than any of the finite sets X_i.

For the next question, you may use without proof the (surprisingly difficult to prove) Cantor-Bernstein theorem:

Theorem 1.7 (Cantor-Bernstein)**.** Suppose that X and Y are sets such that $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$.
(In other words, if there are injective functions $f : X \to Y$ and $g : Y \to X$ then there exists a *bijection* $h : X \to Y$.)

Question 8. For a given set X, inductively define $\mathcal{P}^0(X) = X$, $\mathcal{P}^1(X) = \mathcal{P}(X)$ and

$$\mathcal{P}^{i+1}(X) = \mathcal{P}(\mathcal{P}^i(X)) \ \forall \ i > 1.$$

Let

$$\mathcal{L}(X) = \bigcup_{i \in \mathbb{N}} \mathcal{P}^i(X)$$

Show that $|\mathcal{P}^n(X)| \lneq |\mathcal{L}(X)|$ for all $n \in \mathbb{N}$.

Hint: Suppose for contradiction that there is a bijection $f : \mathcal{L}(X) \to \mathcal{P}^n(X)$ for some n.

Aside: $\mathcal{L}(X)$ is massively larger than X, or even $\mathcal{P}(X)$. See Question 10 for the case of $X = \emptyset$.

Solution: Firstly, the fact that $|\mathcal{P}^n(X)| \leq |\mathcal{L}(X)|$ follows applying the result of the previous example to the particular case here with $X_i = \mathcal{P}^i(X)$. So it remains to show that the inequality is strict.
Suppose for contradiction that we have $|\mathcal{P}^n(X)| = |\mathcal{L}(X)|$ for some n. Then following the hint we want to consider a bijection

$$f : \mathcal{L}(X) \to \mathcal{P}^n(X)$$

In particular, f is injective.
But also, $\mathcal{L}(X)$ contains a copy of $\mathcal{P}^{n+1}(X)$, and so by restricting the domain to $\mathcal{P}^{n+1}(X)$, we obtain an injective function

$$f' : \mathcal{P}^{n+1}(X) \to \mathcal{P}^n(X).$$

That means that $\mathcal{P}^{n+1}(X) \leq \mathcal{P}^n(X)$. But we also know that there is an injective function $g : \mathcal{P}^n(X) \to \mathcal{P}^{n+1}(X)$ which sends every element of $\mathcal{P}^n(X)$ to its singleton set in $\mathcal{P}(\mathcal{P}^n(X))$. Hence by the Cantor-Bernstein theorem, we have $|\mathcal{P}^n(X)| = |\mathcal{P}^{n+1}(X)|$.
But we know that this is false since $\mathcal{P}^{n+1}(X)$ is strictly bigger than $\mathcal{P}^n(X)$, and we obtain the required contradiction.

Question 9. Deduce that there is a set of cardinality larger than any of those in the sequence

$$\aleph_0 = |\mathbb{N}| \lneq |\mathcal{P}(\mathbb{N})| \lneq |\mathcal{P}^2(\mathbb{N})| \lneq \dots$$

These ideas can produce even larger sets. Try to find the biggest one you can!

Solution: We showed that $\mathcal{L}(\mathbb{N})$ has larger cardinality than any of those on this list. We can produce a yet larger one by taking $\mathcal{P}(\mathcal{L}(\mathbb{N}))$ and extend this to a chain

$$|\mathcal{L}(\mathbb{N})| \lneq |\mathcal{P}(\mathcal{L}(\mathbb{N}))| \lneq |\mathcal{P}^2(\mathcal{L}(\mathbb{N}))| \lneq \dots$$

Then we could take the indexed union of these, which would be $\mathcal{L}(\mathcal{L}(\mathbb{N}))$.
But then we could do the same trick as the previous example. Let $\mathcal{L}^1(\mathbb{N}) = \mathcal{L}(\mathbb{N})$ and for each $i > 1$ define $\mathcal{L}^i(\mathbb{N}) = \mathcal{L}(\mathcal{L}^{i-1}(\mathbb{N}))$ We have an indexed union $\bigcup_{i\in\mathbb{N}} \mathcal{L}^i(\mathbb{N})$... and by exactly the same argument as above, its cardinality is larger than any $\mathcal{L}^n(\mathbb{N})$. We could call this set $\mathcal{Q}(\mathbb{N})$.
Then we could try to set up a new indexed union with $X_0 = \mathbb{N}$, $X_1 = \mathcal{L}(\mathbb{N})$, $X_2 = \mathcal{Q}(\mathbb{N})$, $X_3 = ?$ ...

Question 10. Let $Y = \bigcup_{i\in\mathbb{N}} X_i$ be an indexed union with indexing set $\mathbb{N}$ and for which every X_i is finite. Show that Y is countable.

Hint: This is a direct generalization of Question 4. Try to copy the same argument.

Deduce that $|\mathcal{L}(\emptyset)| = \aleph_0$.

Solution: First, we prove that Y is countable. For each $i \in \mathbb{N}$, write $X_i = \{x_{1,i}, \dots, x_{k_i,i}\}$ (we can do this since each X_i is finite). Note that X_i could potentially be empty. We can then list the elements of Y as: $x_{1,1}, \dots, x_{k_1,1}, x_{1,2}, \dots, x_{k_2,2}, x_{1,3}, \dots, x_{k_3,3}, \dots$. Now we apply Theorem 1.9 to conclude that Y is countable.

Now since $\mathcal{L}(\emptyset) = \bigcup_{i\in\mathbb{N}} \mathcal{P}^i(\emptyset)$ and $\mathcal{P}^i(\emptyset)$ is a finite set for all $i \in \mathbb{N}$, we conclude that $\mathcal{L}(\emptyset)$ is countable. Moreover, $\mathcal{L}(\emptyset)$ has infinitely many elements since $|\mathcal{P}^i(\emptyset)| < |\mathcal{L}(\emptyset)|$ for all $i \in \mathbb{N}$, and $\mathcal{P}^i(\emptyset)$ is an infinite strictly increasing sequence of natural numbers. Therefore, $|\mathcal{L}(\emptyset)| = \aleph_0$.