14.6 Tangent Planes & Differentials

Tangent Planes & Normal Lines

Suppose $r = g(t)i + h(t)j + k(t)k$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable fctn f. Then

$$f(g(t), h(t), k(t)) = c$$

Now (as we did in lower dimensional case of 14.5) observe that, differentiating both sides gives us

$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt} c$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) \cdot \left(\frac{dg}{dt} i + \frac{dh}{dt} j + \frac{dk}{dt} k\right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

Therefore,
∇f is orthogonal to the curve's velocity vector (at every point along the curve)

Definitions

The <u>tangent plane</u> at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f\big|_{P_0}$

The <u>normal line</u> of the surface at P_0 is the line through P_0 parallel to $\nabla f \big|_{P_0}$

<u>Tangent Plane</u> to $f(x,y,z) = C$ at P_0

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$$

<u>Normal Line</u> to $f(x,y,z) = C$ at P_0

$$x = x_0 + f_x(P_0)\,t$$
$$y = y_0 + f_y(P_0)\,t$$
$$z = z_0 + f_z(P_0)\,t$$

<u>Example 1</u>:

Find the tangent plane & normal line of the surface $x^2 - xy - y^2 - z = 0$ at the point $(1,1,-1)$

$$f(x,y,z) = x^2 - xy - y^2 - z$$

$f_x = 2x - y$, $f_y = -x - 2y$, $f_z = -1$

$f_x(P_0) = 1$, $f_y(P_0) = -3$, $f_z = -1$

Tangent Plane:

$$(x-1) - 3(y-1) - (z+1) = 0$$
$$x - 3y - z = -1$$

Normal Line:

$x = 1 + t$, $y = 1 - 3t$, $z = -1 - t$

<u>Finding the plane tangent to a smooth surface $z = f(x,y)$ at a point $P_0(x_0, y_0, z_0)$</u>

First observe

$$z = f(x,y)$$
$$f(x,y) - z = 0$$

Thus $z = f(x,y)$ is simply the 0 level surface of the smooth function

$$F(x,y,z) = f(x,y) - z$$

Plane Tangent to a surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example 2

Find the plane tangent to the surface $z = 4x^2 + y^2$ at $(1, 1, 5)$

$f(x, y) = 4x^2 + y^2$

$f_x = 8x \quad , \quad f_y = 2y$

$f_x(1,1) = 8 \quad , \quad f_y(1,1) = 2$

Tangent Plane:

$$8(x-1) + 2(y-1) - (z-5) = 0$$

$$8x + 2y - z = 5$$

Finding parametric equations for the line tangent to the intersection of 2 surfaces at a point P_0

Given 2 surfaces:

$$f(x, y, z) = 0 \quad \& \quad g(x, y, z) = 0$$

that intersect

The line tangent to the curve of intersection at the point $P_0(x_0, y_0, z_0)$ is orthogonal to both ∇f & ∇g at P_0.

Thus, the tangent line is parallel to

$$v = (\nabla f)|_{P_0} \times (\nabla g)|_{P_0}$$

$$= v_1 i + v_2 j + v_3 k$$

Therefore, the tangent line is

$$x = x_0 + v_1 t$$

$$y = y_0 + v_2 t$$

$$z = z_0 + v_3 t$$

Example 3

Find the line tangent to the curve of intersection of the surfaces

$$xyz = 1 \quad \& \quad x^2 + 2y^2 + 3z^2 = 6$$

at the point $(1, 1, 1)$

$f(x, y, z) = xyz - 1$

$g(x, y, z) = x^2 + 2y^2 + 3z^2 - 6$

So,

$$\nabla f = yz\,i + xz\,j + xy\,k$$

$$\nabla g = 2x\,i + 4y\,j + 6z\,k$$

$$(\nabla f)\big|_{(1,1,1)} = i + j + k$$

$$(\nabla g)\big|_{(1,1,1)} = 2i + 4j + 6k$$

$$v = (\nabla f)\big|_{(1,1,1)} \times (\nabla g)\big|_{(1,1,1)}$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2i - 4j + 2k$$

Tangent line:

$$x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t$$

Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction u, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot u)}_{\text{Directional Derivative}} \cdot \underbrace{ds}_{\text{distance increment}}$$

Example 4

Estimate how much the value of

$$f(x,y,z) = e^x \cos(yz)$$

will change if the point $P(x,y,z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2i + 2j - 2k$

First we need the direction of $2i+2j-2k$

$$u = \frac{2i+2j-2k}{\sqrt{(2)^2+(2)^2+(-2)^2}} = \frac{2i+2j-2k}{\sqrt{12}}$$

$$= \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j - \frac{1}{\sqrt{3}} k$$

Now we take the partials of f

$$f_x = e^x \cos(yz)$$
$$f_y = -ze^x \sin(yz)$$
$$f_z = -ye^x \sin(yz)$$

The gradient of f at $(0,0,0)$ is

$$(\nabla f)\Big|_{(0,0,0)} = e^0 \cos(0)\, i - 0j - 0k$$
$$= i$$

Now,

$$\nabla f\Big|_{(0,0,0)} \cdot u = \frac{1}{\sqrt{3}}$$

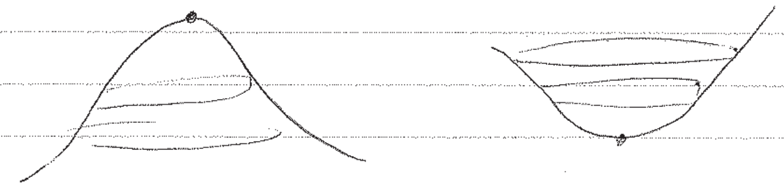
$$df = \frac{1}{\sqrt{3}}(0.1) = \frac{1}{10\sqrt{3}}$$

14.7 Extreme Values & Saddle Points

Definition

Let $f(x,y)$ be defined on a region R containing the point (a,b) Then

1. $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ $\forall (x,y)$ in an open disk centered at (a,b)
2. $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ $\forall (x,y)$ in an open disk centered at (a,b)



Note:

Both of the above are called local extrema OR relative extrema

Theorem 10 1st Derivative Test

If $f(x,y)$ has a local max or min value at an interior point (a,b) of its domain & the 1st partials exist then

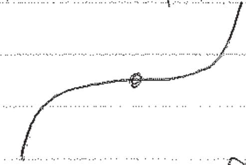
$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

Definition:

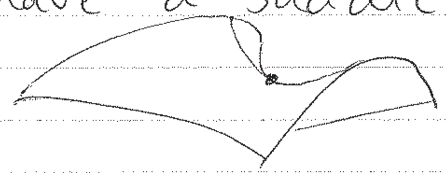
An interior point of the domain of a fctn $f(x,y)$, where both f_x & f_y are 0 or where 1 or both do NOT exist is a critical point of f

Recall:

- A function of 1 variable sometimes has critical points which are NOT local extrema (ie: inflection points)



- Similarly, a function of 2 variables may have a saddle point



Definition

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if ANY/EVERY open disk centered at (a,b) contains points (x,y) with $f(x,y) > f(a,b)$ & points (x,y) with $f(x,y) < f(a,b)$

Examples!

Find the local extreme values (if any) of the following

a) $f(x,y) = x^4 + 3y^2$

The domain here is $\mathbb{R}^2$ & we can take partial derivatives everywhere

$f_x = 4x^3$ & $f_y = 6y$

So the only possibility is the origin.

Note: At the origin $f(0,0) = 0$, but everywhere else $f(x,y) > 0$

So, the origin is a local minima

b) $f(x,y) = y^4 - x^4 + 7$

Again the domain is $\mathbb{R}^2$ & we can take partial derivatives everywhere

$f_x = -4x^3$ & $f_y = 4y^3$
So again the only possibility is the origin
Note: At the origin $f(0,0) = 7$, BUT along the x-axis (where $y=0$), f has the value $f(x,0) = -x^4 + 7 < 7$
AND along the y-axis (where $x=0$) f has the value $f(0,y) = y^4 + 7 > 7$
Thus since every open disk in the xy-plane centered at $(0,0)$ contains points where $f(x,y) > 7$ & points where $f(x,y) < 7$, the function MUST have a saddle point at the origin

<u>Theorem 11</u> 2nd Derivative Test
Suppose that $f(x,y)$ & its 1st & 2nd derivatives are continuous in a disk centered at (a,b) & $f_x(a,b) = 0 = f_y(a,b)$. Then

1. (a,b) is a <u>local maximum</u> of f if $f_{xx} < 0$ & $f_{xx}f_{yy} - f_{xy}^2 > 0$
2. (a,b) is a <u>local minimum</u> of f if $f_{xx} > 0$ & $f_{xx}f_{yy} - f_{xy}^2 > 0$
3. (a,b) is a <u>saddle point</u> of f if $f_{xx}f_{yy} - f_{xy}^2 < 0$
4. If $f_{xx}f_{yy} - f_{xy}^2 = 0$, then the test is <u>Inconclusive</u>

<u>Note</u>:
$f_{xx}f_{yy} - f_{xy}^2$ is called the <u>discriminant</u> or <u>Hessian</u> of f
- When $f_{xx}f_{yy} - f_{xy}^2 > 0$ the surface curves the same way in all directions

Remember the Hessian:

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

<u>Example 2</u>

Find the local extreme values of

$$f(x,y) = 3 + 2x + 2y - 2x^2 - 2xy - y^2$$

The domain is $\mathbb{R}^2$ & f is differentiable everywhere. First we look for extreme points

$$f_x = 2 - 4x - 2y = 2(1 - 2x - y)$$
$$f_y = 2 - 2x - 2y = 2(1 - x - y)$$

Setting $f_x = 0 = f_y$, we solve for x & y

$$y = 1 - x, \quad x = 0 \implies y = 1$$

Thus, $(0,1)$ is the ONLY possible point f can take on an extreme value.

Now, we look at 2nd derivatives:

$$f_{xx} = -4, \quad f_{yy} = -2, \quad f_{xy} = -2$$

Note:

$$f_{xx}f_{yy} - f_{xy}^2 = (-4)(-2) - (-2)^2 = 4 > 0$$

AND

$$f_{xx} = -4 < 0$$

Therefore f has a local maximum at $(0,1)$.
The value of f at this point is

$$f(0,1) = 4$$

<u>Example 3</u>

Find all local max, min, or saddle points of

$$f(x,y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$$

Again domain is $\mathbb{R}^2$ & f is differentiable everywhere.

$$f_x = 6x^2 - 18x = 6(x^2 - 3x)$$
$$f_y = 6y^2 + 6y - 12 = 6(y^2 + y - 2)$$

$f_x = 6x(x-3) \Longrightarrow x=0\ ;\ x=3$

$f_y = 6(y+2)(y-1) \Longrightarrow y=-2\ ,\ y=1$

So we have 4 possibilities

$(0,-2)\quad (0,1)\quad (3,-2)\quad (3,1)$

Now we take 2nd derivatives:

$f_{xx} = 12x - 18 \qquad f_{yy} = 12y + 6$

$f_{xy} = 0$

Test each point now:

$(0,-2)$:

$f_{xx}f_{yy} - f_{xy}^2 = (-18)(-18) - 0 > 0$

$f_{xx} < 0$

Local max.

$(0,1)$:

$f_{xx}f_{yy} - f_{xy}^2 = (-18)(18) - 0 < 0$

Saddle Point

$(3,-2)$:

$f_{xx}f_{yy} - f_{xy}^2 = 18(-18) - 0 < 0$

Saddle Point

$(3,1)$:

$f_{xx}f_{yy} - f_{xy}^2 = 18(18) - 0 > 0$

$f_{xx} > 0$

Local min

Absolute Max/Min on Closed Bounded Region

1. Use 2nd Derivative test to get local extrema
2. Use Domain to list all Boundary Points
3. Look through the lists for max/min of f when you combine the lists from 1. & 2. whichever point gives you the max/min value is the Absolute Max/Min in the region R

Example 4

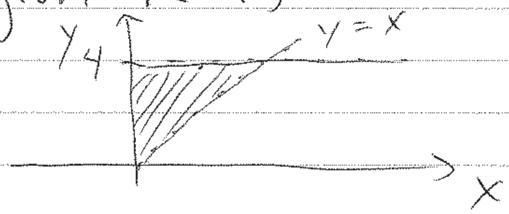
Find the absolute max & min of the function

$$f(x,y) = x^2 - xy + y^2 + 1$$

on the closed triangular plate in the 1st quadrant bounded by the lines

$x=0$, $y=4$, $y=x$

So the region R is



1. Look for local extrema

$f_x = 2x - y \qquad f_y = -x + 2y$

Only possibility is $(0,0)$ NOT interior

2. Look at boundary points.

$x=0$:

$f(0,y) = y^2 + 1$

So extreme values occur at

$2y = 0 \implies y = 0$

$f(0,0) = 1$

$y = 4$

$f(x,4) = x^2 - 4x + 17$

Extreme values occur at

$2x - 4 = 0 \implies x = 2$

$f(2,4) = 13$

$y = x$

$f(x,x) = x^2 + 1$

Extreme values occur at

$2x = 0 \implies x = 0$

$f(0,0) = 1$

Thus we have absolute min at $(0,0)$ & absolute max at $(2,4)$