

5.1 The Remainder and Factor Theorems; Synthetic Division

In this section you will learn to:

- understand the definition of a zero of a polynomial function
- use long and synthetic division to divide polynomials
- use the remainder theorem
- use the factor theorem

Example 1: Use long division to find the quotient and the remainder: $5593 \div 27$

Steps for Long Division:

- 1.
- 2.
- 3.
- 4.

Example 2: Use the “Steps for Long Division” to divide each of the polynomials below.

$$x - 5 \overline{) x^2 - 2x - 35}$$

$$(7 - 11x - 3x^2 + 2x^3) \div (x - 3)$$

Example 3: Check your answer for the division problems in Example 2.

The Division Algorithm: If $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and degree $d(x) < \text{degree } f(x)$,

then

$$\overline{) }$$

$$f(x) = d(x) \cdot q(x) + r(x)$$

If $r(x) = 0$ then $d(x)$ and $q(x)$ are **factors** of $f(x)$.

Example 4: Perform the operation below. Write the remainder as a rational expression (remainder/divisor).

$$\frac{2x^5 - 8x^4 + 2x^3 + x^2}{2x^3 + 1}$$

Synthetic Division – Generally used for “short” division of polynomials when the divisor is in the form $x - c$. (Refer to page 506 in your textbook for more examples.)

Example 5: Use both long and short (synthetic) division to find the quotient and remainder for the problem below.

$$(2x^3 - 11x + 7) \div (x - 3)$$

Example 6: Divide $\frac{x^3 + 8}{x + 2}$ using synthetic division.

Example 7: Factor $x^3 + 8$ over the real numbers. (Hint: Refer to Example 6.)

Remainder Theorem	Factor Theorem
If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.	<p>Let $f(x)$ be a polynomial.</p> <p>If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$.</p> <p>If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.</p> <p>If $(x - c)$ is a factor of $f(x)$ or if $f(c) = 0$, then c is called a zero of $f(x)$.</p>

Example 8: $f(x) = 3x^3 + 4x^2 - 5x + 7$. Find $f(-4)$ using

(a) synthetic division.

(b) the Remainder Theorem.

Example 9: Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that -2 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

5.1 Homework Problems:

For Problems 1-5, use long division to find each quotient, $q(x)$, and remainder, $r(x)$.

1. $(x^2 - 2x - 15) \div (x - 5)$

2. $(x^3 + 5x^2 + 7x + 2) \div (x + 2)$

3. $(6x^3 + 7x^2 + 12x - 5) \div (3x - 1)$

4. $\frac{x^4 - 81}{x - 3}$

5. $\frac{18x^4 + 9x^3 + 3x^2}{3x^2 + 1}$

For Problems 6 – 11, divide using synthetic division.

6. $(2x^2 + x - 10) \div (x - 2)$

7. $(5x^3 - 6x^2 + 3x + 11) \div (x - 2)$

8. $(x^2 - 5x - 5x^3 + x^4) \div (5 + x)$

9. $\frac{x^7 + x^5 - 10x^3 + 12}{x + 2}$

10. $\frac{x^4 - 256}{x - 4}$

11. $\frac{x^5 - 2x^4 - x^3 + 3x^2 - x + 1}{x - 2}$

For Problems 12 – 16, use synthetic division and the Remainder Theorem to find the indicated function value.

12. $f(x) = x^3 - 7x^2 + 5x - 6$; $f(3)$

13. $f(x) = 4x^3 + 5x^2 - 6x - 4$; $f(-2)$

14. $f(x) = 2x^4 - 5x^3 - x^2 + 3x + 2$; $f\left(-\frac{1}{2}\right)$

15. $f(x) = 6x^4 + 10x^3 + 5x^2 + x + 1$; $f\left(-\frac{2}{3}\right)$

16. Use synthetic division to divide $f(x) = x^3 - 4x^2 + x + 6$ by $x + 1$. Use the result to find all zeros of f .

17. Solve the equation $2x^3 - 5x^2 + x + 2 = 0$ given that 2 is a zero of $f(x) = 2x^3 - 5x^2 + x + 2$.

18. Solve the equation $12x^3 + 16x^2 - 5x - 3 = 0$ given that $-\frac{3}{2}$ is a zero (root).

5.1 Homework Answers: 1. $q(x) = x + 3$ 2. $q(x) = x^2 + 3x + 1$ 3. $q(x) = 2x^2 + 3x + 5$

4. $q(x) = x^3 + 3x^2 + 9x + 27$ 5. $q(x) = 6x^2 + 3x - 1$; $r(x) = -3x + 1$ 6. $q(x) = 2x + 5$

7. $q(x) = 5x^2 + 4x + 11$; $r(x) = 33$ 8. $q(x) = x^3 - 10x^2 + 51x - 260$; $r(x) = 1300$

9. $q(x) = x^6 - 2x^5 + 5x^4 - 10x^3 + 10x^2 - 20x + 40$; $r(x) = -68$ 10. $q(x) = x^3 + 4x^2 + 16x + 64$

11. $q(x) = x^4 - x^2 + x + 1$; $r(x) = 3$ 12. -27 13. -4 14. 1 15. $\frac{7}{9}$

16. $x^2 - 5x + 6$; $x = -1, 2, 3$ 17. $\left\{-\frac{1}{2}, 1, 2\right\}$ 18. $\left\{-\frac{3}{2}, -\frac{1}{3}, \frac{1}{2}\right\}$

5.3 Roots of Polynomial Equations

In this section you will learn to:

- find zeros of polynomial equations
- solve polynomial equations with real and imaginary zeros
- find possible rational roots of polynomial equations
- understand properties of polynomial equations
- use the Linear Factorization Theorem

Zeros of Polynomial Functions are the values of x for which $f(x) = 0$.

(Zero = Root = Solution = x -intercept (if the zero is a real number))

Example 1: Consider the polynomial that has exactly two zeros: 3 and $\frac{1}{2}$

- (a) How many polynomials have such zeros?
- (b) Find a polynomial that has a leading coefficient of 1 that has such zeros.
- (c) Find a polynomial, with integral coefficients, that has such zeros.

If the same factor $(x - r)$ occurs k times, then the zero r is called a zero with **multiplicity k** .

Even Multiplicity → Graph **touches** x -axis and turns around.

Odd Multiplicity → Graph **crosses** x -axis.

Example 2: Find all of the (real) zeros for each of the polynomial functions below. Give the multiplicity of each zero and state whether the graph crosses the x -axis or touches (and turns at) the x -axis at each zero. Use this information and the Leading Coefficient Test to sketch a graph of each function

(a) $f(x) = x^3 + 2x^2 - 4x - 8$

(b) $f(x) = -x^4 + 4x^2$

(c) $g(x) = -x^4 + 4x^3 - 4x^2$

The Rational Zero Theorem: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ has **integer** coefficients then $\frac{p}{q}$ (reduced to lowest terms) is a rational zero of $f(x)$, where p is a factor of the constant term, a_0 , and q is a factor of the leading coefficient, a_n .

Example 3: List all **possible** rational zeros of the polynomials below. (Refer to Rational Zero Theorem on

Page 1 of this handout.)

(a) $f(x) = -x^5 + 7x^2 - 12$ Possible Rational Zeros: _____

(b) $p(x) = 6x^3 - 8x^2 - 8x + 8$ Possible Rational Zeros: _____

Example 4: Find all zeros of $f(x) = 2x^3 - 5x^2 + x + 2$.

Example 5: Solve $x^4 - 8x^3 + 64x - 105 = 0$.

Linear Factorization Theorem:

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$$f(x) = a_n (x - c_1)(x - c_2) \dots (x - c_n), \text{ where } c_1, c_2, c_3, \dots, c_n \text{ are complex numbers.}$$

Example 6: Find all complex zeros of $f(x) = 2x^4 + 3x^3 + 3x - 2$, and then write the polynomial $f(x)$ as a **product of linear factors**.

$$f(x) = \underline{\hspace{10cm}}$$

Properties of Polynomial Equations:

Given the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$.

1. If a polynomial equation is of degree n , then counting multiple roots (multiplicities) separately, the equation has n roots.
2. If $a + bi$ is a root of a polynomial equation ($b \neq 0$), then the imaginary number $a - bi$ is also a root. In other words, imaginary roots, if they exist, occur in **conjugate pairs**.

Example 7: Find all zeros of $f(x) = x^4 - 4x^2 - 5$. (Hint: Use factoring techniques from Chapter 1.) Write $f(x)$ as a product of linear factors.

$$f(x) = \underline{\hspace{10cm}}$$

Example 8: Find a third-degree polynomial function, $f(x)$, with real coefficients that has 4 and $2i$ as zeros and such that $f(-1) = 50$.

Step 1: Use the zeros to find the factors of $f(x)$.

Step 2: Write as a linear factorization, then expand/multiply.

Step 3: Use $f(-1) = 50$ to substitute values for x and $f(x)$.

Step 4: Solve for a_n .

Step 5: Substitute a_n into the equation for $f(x)$ and simplify.

Step 6: Use your calculator to check.

5.3 Homework Problems:

For Problems 1 – 4, use the Rational Zero Theorem to list all possible rational zeros for each function.

1. $f(x) = x^3 + 3x^2 - 6x - 8$

2. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$

3. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$

4. $f(x) = 4x^5 - 8x^4 - x + 2$

For Problems 5 – 8, find the zeros for the given functions.

5. $f(x) = x^3 - 2x^2 - 11x + 12$

6. $f(x) = 2x^3 - 5x^2 + x + 2$

7. $f(x) = 2x^3 + x^2 - 3x + 1$

8. $f(x) = x^3 - 4x^2 + 8x - 5$

For Problems 9 – 12, solve each of the given equations.

9. $x^3 - 2x^2 - 7x - 4 = 0$

10. $x^3 - 5x^2 + 17x - 13 = 0$

11. $2x^3 - 5x^2 - 6x + 4 = 0$

12. $x^4 - 2x^2 - 16x - 15 = 0$

For Problems 13-16, find an n th degree polynomial function, $f(x)$, with real coefficients that satisfies the given conditions.

13. $n = 3$; 1 and $5i$ are zeros; $f(-1) = -104$

14. $n = 4$; 2, -2, and i are zeros; $f(3) = -150$

15. $n = 3$; 6 and $-5 + 2i$ are zeros; $f(2) = -636$

16. $n = 4$; i and $3i$ are zeros; $f(-1) = 20$

5.3 Homework Answers: 1. $\pm 1, \pm 2, \pm 4, \pm 8$ 2. $\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}$

3. $\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$ 4. $\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$ 5. -3, 1, 4 6. $-\frac{1}{2}, 1, 2$

7. $\frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2}$ 8. $1, \frac{3 \pm i\sqrt{11}}{2}$ 9. $\{-1, 4\}$ 10. $\{1, 2 \pm 3i\}$ 11. $\left\{\frac{1}{2}, 1 \pm \sqrt{5}\right\}$

12. $\{-1, 3, -1 \pm 2i\}$ **13.** $f(x) = 2x^3 - 2x^2 + 50x - 50$ **14.** $f(x) = -3x^4 + 9x^2 + 12$

15. $f(x) = 3x^3 + 12x^2 - 93x - 522$ **16.** $f(x) = x^4 + 10x^2 + 9$