3 Shortcuts to Differentiation

3.1 Derivative Formulas for Powers and Polynomials

*** Derivative of a Constant Function**

If $f(x) = k$ and k is a constant, then $f'(x) = 0$.

Example 1 *Find the derivative of* $f(x) = 5$.

*** Derivative of a Linear Function**

If $f(x) = b + mx$, then $f'(x) = \text{slope} = m$.

Example 2 *Find the derivative of* $f(x) = 5 - \frac{3}{2}x$.

*** Powers of x**

The Power Rule
For any constant real number n,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 3 *Find the derivative of*

(a) $y = x^{12}$ *(b)* $y = x^{-12}$ *(c)* $y = x^{\frac{4}{3}}$

(d) $y = \frac{1}{x^4}$ *(e)* $y = \sqrt{x}$ *(f)* $y = \sqrt{\frac{1}{x^3}}$

*** Derivative of a Constant Times a Function**

If c is a constant, then

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

*** Derivative of Sums and Differences**

$$\begin{aligned} \frac{d}{dx}[f(x)+g(x)] &= f'(x)+g'(x) \\ \frac{d}{dx}[f(x)-g(x)] &= f'(x)-g'(x) \end{aligned}$$

*** Derivative of Polynomials**

Example 4 *Find the derivative of*

(a) $A(t) = 3t^5$

(b) $r(p) = p^5 + p^3$

(c) $f(x) = 5x^2 - 7x^3$

(d) $g(t) = \dfrac{t^2}{4} + 3$

(e) $f(x) = 6x^3 + 4x^2 - 2x$

Example 5 *Find the derivative of* $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2})$.

*** Using the Derivative Formulas**

Example 6 *Find an equation for the tangent line at* $x = 1$ *to the graph of*

$$y = x^3 + 2x^2 - 5x + 7.$$

Sketch the graph of the curve and its tangent line on the same axes.

Example 7 *The number,* N*, of acres of harvested land in a region is given by*

$$N = f(t) = 120\sqrt{t},$$

where t *is the number of years since farming began in the region. Find* $f(10)$, $f'(10)$*, and the relative rate of change* f'/f *at* $t = 10$*. Interpret your answers in terms of harvested land.*

Example 8 *If* $f(t) = t^4 - 3t^2 + 5t$*, find* $f'(t)$ *and* $f''(t)$*.*

Example 9 *The cost (in dollars) of producing q items is given by* $C(q) = 0.08q^3 + 75q + 1000$*.*

(a) *Find the marginal cost function.*

(b) *Find* $C(50)$ *and* $C'(50)$*. Give units and interpret your answers.*

Example 10 *The revenue (in dollars) from producing q units of a product is given by* $R(q) = 1000q - 3q^2$*. Find* $R(125)$ *and* $R'(125)$*. Given units and interpret your answers.*

3.2 Exponential and Logarithmic Functions

*** The Derivative of** e^x

$$\frac{d}{dx}(e^x) = e^x$$

*** The Exponential Rule**

For any positive constant a,

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

Example 1 *Find the derivative of* $f(x) = 2 \cdot 3^x + 5e^x$.

*** The Derivative of** e^{kt}

If k is a constant,

$$\frac{d}{dt}(e^{kt}) = ke^{kt}.$$

Example 2 *Find the derivative of* $P = 5 + 3x^2 - 7e^{-0.2x}$.

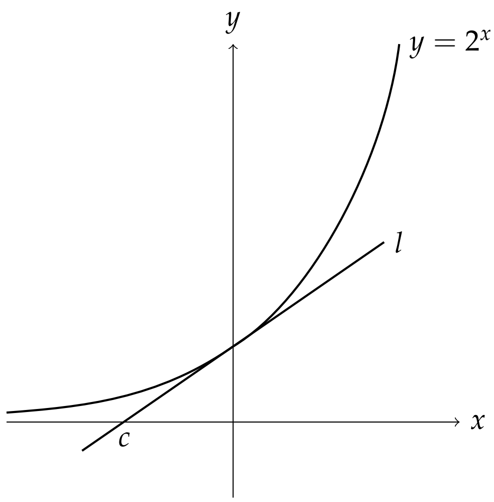
*** The Derivative of** $\ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example 3 *Differentiate* $y = 5\ln t + 7e^t - 4t^2 + 12$.

*** Using the Derivative Formulas**

Example 4 *Find the value of c in the following figure, where the line l tangent to the graph of* $y = 2^x$ *at* $(0, 1)$ *intersects the x-axis.*



Example 5 *Given* $f(x) = \ln x$.

(a) Find the equation of the tangent line to the graph of $f(x) = \ln x$ *at* $x = 1$.

(b) Use (a) to approximate values for $\ln(1.1)$ *and* $\ln(2)$.

(c) Using a graph, explain whether the approximate values are smaller or larger than the true values.

Example 6 *The Population of Nevada, P, in millions, can be approximated by*

$$P = 2.020(1.036)^t,$$

where t is years since the start of 2000. At what rate was the population growing at the beginning of 2009? Give units with your answer.

Example 7 *Suppose \$1000 is deposited into a bank account that pays 8% annual interest, compounded continuously.*

(a) Find a formula $f(t)$ for the balance t years after the initial deposit.

(b) Find $f(10)$ and $f'(10)$ and explain what your answers mean in terms of money.

3.3 The Chain Rule

*** Composite Functions**

Example 1 *Use a new variable z for the inside function to express each of the following as a composite function:*
(a) $y = \ln(3t)$ *(b)* $P = e^{-0.03t}$ *(c)* $w = 5(2r+3)^2$

*** The Derivative of Composite Functions**

The Chain Rule
If $y = f(z)$ and $z = g(t)$ are differentiable, then the derivative of $y = f(g(t))$ is given by

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}.$$

In words, the derivative of a composite function is the derivative of the outside function times the derivative of the inside function:

$$\frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t).$$

Example 2 *Find the derivative of the following functions:*

(a) $y = (4t^2+1)^7$

(b) $P = e^{3t}$

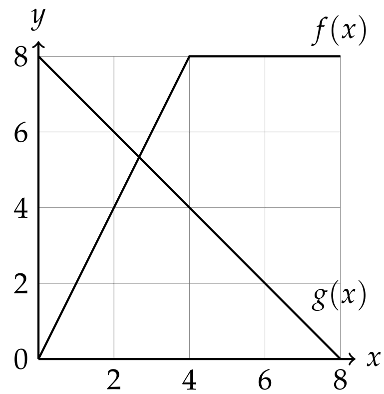
The Chain Rule
If z is a differentiable function of t, then

$$\begin{aligned}\frac{d}{dt}(z^n) &= nz^{n-1}\frac{dz}{dt},\\ \frac{d}{dt}(e^z) &= e^z\frac{dz}{dt},\\ \frac{d}{dt}(\ln z) &= \frac{1}{z}\frac{dz}{dt}.\end{aligned}$$

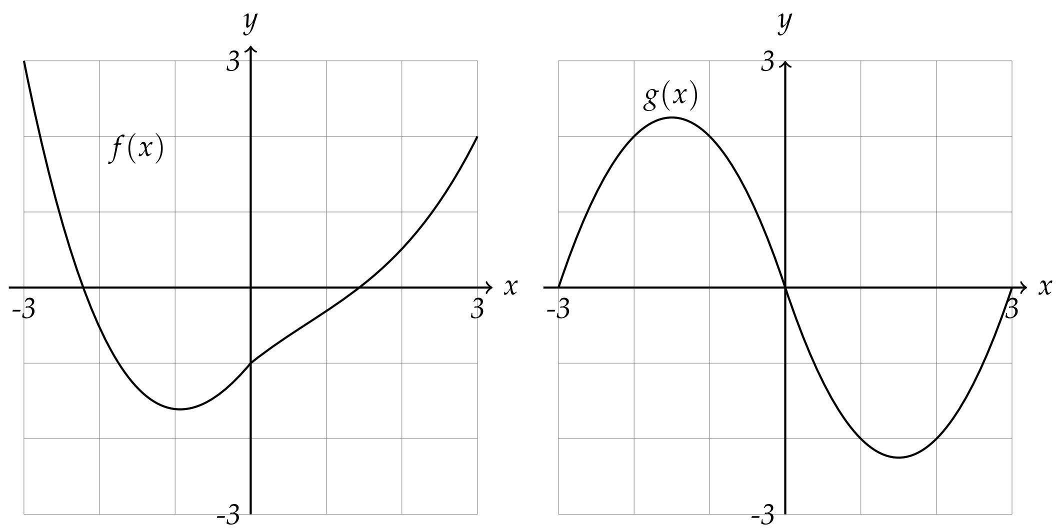
Example 3 *Differentiate (a)* $(3t^3 - t)^5$ *(b)* $\ln(q^2+1)$ *(c)* e^{-x^2}.

Example 4 *Differentiate (a)* $(x^2+4)^3$ *(b)* $5\ln(2t^2+3)$ *(c)* $\sqrt{1+2e^{5t}}$

Example 5 *Let $h(x) = f(g(x))$ and $k(x) = g(f(x))$. Use the following figure to estimate (a) $h'(1)$ and (b) $k'(2)$.*



Example 6 *Let $h(x) = f(g(x))$ and $k(x) = g(f(x))$. Use the following figure to estimate (a) $h'(1)$ and (b) $k'(3)$.*



Example 7 *If you invest P dollars in a bank account at an annual interest rate of r%, then after t year you will have B dollars, where*

$$B = P\left(1 + \frac{r}{100}\right)^t.$$

(a) Find dB/dt, assuming P and r are constant. In terms of money, what does dB/dt represent?

(b) Find dB/dr, assuming P and t are constant. In terms of money, what does dB/dr represent?

*** Relative Rates and Logarithms**

For any positive function $f(t)$,

$$\text{Relative rate of change of } f(t) = \frac{d}{dt}(\ln f(t)).$$

Example 8 *Find the relative rate of change of $f(t)$ for (a) $f(t) = 6.8e^{-0.5t}$ and (b) $f(t) = 4.5t^{-4}$.*

Example 9 *The surface area S of a mammal, in cm^2, is a function of the body mass, M, of the mammal, in kilograms, and is given by $S = 1095 \cdot M^{2/3}$. Find the relative rate of change of S with respect to M and evaluate for a human with body mass 70 kilograms. Interpret your answer.*

3.4 The Product and Quotient Rules

*** The Product Rule**

The Product Rule If $u = f(x)$ and $v = g(x)$ are differentiable functions, then

$$(fg)' = f'g + fg'.$$

The product rule can also be written

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

In words: The derivative of a product is the derivative of the first times the second, plus the first times the derivative of the second.

Example 1 *Differentiate (a)* xe^x *(b)* $t \ln t$ *(c)* $(t^3 + 5t)(t^2 - 7t + 2)$.

Example 2 *Differentiate (a)* $x^2 e^{2x}$ *(b)* $t^3 \ln(5t + 1)$ *(c)* $(3x^2 + 5x)e^{x^2}$ *(d)* $(te^{3t} + e^{5t})^9$.

*** The Quotient Rule**

The Quotient Rule If $u = f(x)$ and $v = g(x)$ are differentiable functions, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

The quotient rule can also be written

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - u\cdot \frac{dv}{dx}}{v^2}.$$

In words: The derivative of a quotient is the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all over the denominator squared.

Example 3 *Differentiate (a)* $\dfrac{5x-2}{x^3+1}$ *(b)* $\dfrac{e^x}{1+e^x}$ *(c)* $\dfrac{e^x}{x^2+\ln x}$.

Example 4 *If* $f(x) = (3x+8)(2x-5)$, *find* $f'(x)$ *and* $f''(x)$.

Example 5 *Assume that a, b, c, d are constants, find the derivative of (a)* $\dfrac{ax+b}{cx+d}$ *(b)* axe^{-bx} *(c)* $(ax^2+b)^3$ *(d)* $e^{xe^{-2x}}$.

Example 6 *Find the equation of the tangent line to the graph of* $f(x) = x^2e^{-x}$ *at* $x = 0$.

Example 7 *Find the equation of the tangent line to the graph of* $f(x) = \dfrac{2x-5}{x+1}$ *at* $x = 0$.

3.5 Derivatives of Periodic Functions

*** The Sine and Cosine**

For x in radians,

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x,\\ \frac{d}{dx}(\cos x) &= -\sin x.\end{aligned}$$

Example 1 *Differentiate (a)* $5\sin t - 8\cos t$ *and (b)* $5 - 3\sin x + x^3$.

If z is a differentiable function of t, then

$$\begin{aligned}\frac{d}{dt}(\sin z) &= \cos z\frac{dz}{dt},\\ \frac{d}{dt}(\cos z) &= -\sin z\frac{dz}{dt}.\end{aligned}$$

If k is a constant, then

$$\begin{aligned}\frac{d}{dt}(\sin kt) &= k\cos kt,\\ \frac{d}{dt}(\cos kt) &= -k\sin kt.\end{aligned}$$

Example 2 *Differentiate (a)* $\sin(t^2)$ *(b)* $5\cos(2t)$ *(c)* $6\sin(2t) + \cos(4t)$ *(c)* $2t^3\sin(3t)$
(e) $\dfrac{e^{t^2} + t}{\sin 2t}$

Example 3 *Find the equation of the tangent line to the graph of* $y = \sin x$ *at* $x = \pi$.

Example 4 *Is the graph of* $y = \sin(x^4)$ *increasing or decreasing when* $x = 10$*? Is it concave up or concave down?*