5.5 The Fundamental Theorem of Calculus

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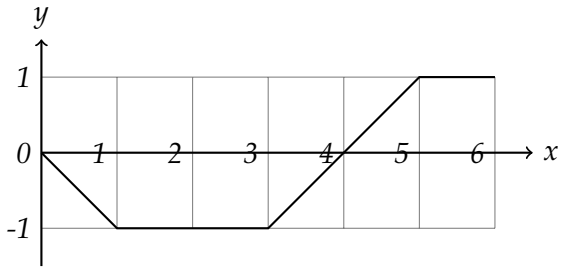
If $F'(t)$ is continuous for $a \leq t \leq b$, then

$$\int_a^b F'(t)dt = F(b) - F(a).$$

In words: The definite integral of the derivative of a function gives the total change in the function.

Example 1 *The graph of a derivative $f'(x)$ is shown in the following figure.*



Fill in the table of values for $f(x)$ given that $f(3) = 2$.

x	0	1	2	3	4	5	6
$f(x)$				2			

*** Marginal Cost and Change in Total Cost**

If $C'(q)$ is a marginal cost function and $C(0)$ is the fixed cost,

$$\begin{aligned} \text{Cost to increase production from } a \text{ units to } b \text{ units} &= C(b) - C(a) = \int_a^b C'(q)dq \\ \text{Total variable cost to produce } b \text{ units} &= \int_0^b C'(q)dq \\ \text{Total cost of producing } b \text{ units} &= \text{Fixed cost } + \text{Total variable cost} \\ &= C(0) + \int_0^b C'(q)dq \end{aligned}$$

Example 2 *The total cost in dollars to produce q units of a product is $C(q)$. Fixed costs are \$20,000. The marginal cost is*

$$C'(q) = 0.005q^2 - q + 56.$$

(a) On a graph of $C'(q)$, illustrate graphically the total variable cost of producing 150 units.

(b) Estimate $C(150)$, the total cost to produce 150 units.

(c) Find the value of $C'(150)$ and interpret your answer in terms of costs of production.

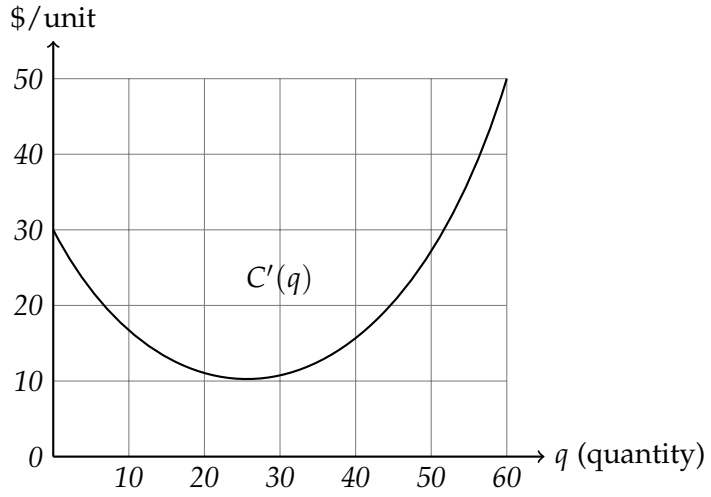
(d) Use parts (b) and (c) to estimate $C(151)$.

Example 3 *A marginal cost function $C'(q)$ is given in the following figure. If the fixed costs are \$10,000, estimate:*

(a) The total cost to produce 30 units.

(b) The additional cost if the company increases production from 30 units to 40 units.

(c) The value of $C'(25)$. Interpret your answer in terms of costs of production.



Example 4 *The marginal cost $C'(q)$ (in dollars per unit) of producing q units is given in the following table.*

q	0	100	200	300	400	500	600
$C'(q)$	25	20	18	22	28	35	45

(a) If fixed cost is \$10,000, estimate the total cost of producing 400 units.

(b) How much would the total cost increase if production were increased one unit, to 401 units?

Example 5 *The marginal cost function of producing q mountain bikes is*

$$C'(q) = \frac{600}{0.3q + 5}.$$

(a) If the fixed cost in producing the bicycle is \$2000, find the total cost to produce 30 bicycles.

(b) If the bikes are sold for \$200 each, what is the profit (or loss) on the first 30 bicycles?

(c) Find the marginal profit on the 31st bicycle.

Focus on Theory

*** The Second Fundamental Theorem of Calculus**

Second Fundamental Theorem of Calculus
If f is a continuous function on an interval, and if a is any number in that interval, then the function G defined on the interval by

$$G(x) = \int_a^x f(t)dt$$

has derivative f; that is, $G'(x) = f(x)$.

Example 1 *Suppose that* $G(x) = \int_3^x \frac{\sin\sqrt{t}}{t+6}dt$. *Find*

(a) $G'(x)$.

(b) $G(0)$.

(c) $G(7)$.

(d) $G'(2)$.

Example 2 *Let* $F(b) = \int_0^b 2^x dx$.

(a) What is $F(0)$*?*

(b) Does the value of F increase or decrease as b increases? Assume $b \geq 0$.

(c) Estimate $F(1), F(2), F(3)$.

Example 3 *For x =0, 0.5, 1.0, 1.5, and 2.0, make a table of values for $I(x) = \int_0^x \sqrt{t^4+1}dt$.*

*** Properties of the Definite Integral**

Sums and Multiples of Definite Integrals
If a, b, and c are any numbers and f and g are continuous functions, then

$$\begin{aligned}\int_a^c f(x)dx + \int_c^b f(x)dx &= \int_a^b f(x)dx,\\ \int_a^b (f(x)+g(x))dx &= \int_a^b f(x)dx + \int_a^b g(x)dx,\\ \int_a^b cf(x)dx &= c\int_a^b f(x)dx.\end{aligned}$$

Example 4 *Let $\int_a^b f(x)dx = 8$, $\int_a^b (f(x))^2dx = 12$, $\int_a^b g(t)dt = 2$, and $\int_a^b (g(t))^2dt = 3$. Find the following integrals.*

(a) $\int_a^b (f(x)+g(x))dx$

(b) $\int_a^b ((f(x))^2 - (g(x))^2)dx$

(c) $\int_a^b (f(x))^2dx - (\int_a^b f(x)dx)^2$

(d) $\int_a^b cf(z)dz$

Example 5 *Given $\int_{-4}^6 f(x)dx = 12$, $\int_{-15}^6 g(x)dx = 5$, and $\int_{-4}^6 g(x)dx = -18$. Find the following integrals.*

(a) $\int_{-4}^6 [3f(x) - \frac{5}{6}g(x)]dx$

(b) $\int_{-15}^{-4} g(x)dx$