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INVARIANT FOLIATIONS FOR RANDOM DYNAMICAL SYSTEMS

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Abstract. We prove the existence of invariant foliations of stable and unstable manifolds of a normally hyperbolic random invariant manifold. The normally hyperbolic random invariant manifold referred to here is that which was shown to exist in a previous paper when a deterministic dynamical system having a normally hyperbolic invariant manifold is subjected to a small random perturbation.

1. **Introduction.** This work is the second step in a program to build a geometric singular perturbation theory under stochastic perturbations. In the first step of the program, [15], we proved the persistence of a deterministic normally hyperbolic invariant manifold under a random perturbation, obtaining a normally hyperbolic random invariant manifold. We also established the existence of its random stable and unstable manifolds. In the current paper, we prove that there exist invariant foliations of these random stable and unstable manifolds with the base points of the fibers on the random normally hyperbolic invariant manifold. The fibers of the foliations vary measurably with respect to the random parameter and smoothly with respect to the base point.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\theta^t)_{t\in\mathbb{R}}$ be a measurable P-measure preserving dynamical system on Ω. The system $(\Omega, \mathcal{F}, P, \theta^t)$ is called a metric dynamical system. A random dynamical system (or a cocycle) on space $\mathbb{R}^n$ over the metric dynamical system θ^t is a measurable map

$$\phi(\cdot,\cdot,\cdot) : \mathbb{R} \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

such that the map $\phi(t, \omega) := \phi(t, \omega, \cdot)$ forms a cocycle over θ^t:

$$\phi(0, \omega) = Id, \quad \text{ for all } \omega \in \Omega,$$

$$\phi(t + s, \omega) = \phi(t, \theta^s\omega)\phi(s, \omega), \quad \text{ for all } t, s \in \mathbb{R}, \quad \omega \in \Omega.$$

When $\phi(\cdot, \omega, \cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for each $\omega \in \Omega$, $\phi(t, \omega, x)$ is called a continuous random dynamical system. A continuous RDS ϕ is called a smooth RDS of class C^k, $k \geq 1$ if for each $(t, \omega) \in \mathbb{R} \times \Omega$, $\phi(t, \omega, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is C^k–smooth. More precisely, $\phi(t, \omega, x)$ is k times differentiable in x and the derivatives are continuous in (t, x) for each $\omega \in \Omega$.

Random dynamical systems arise in the modeling of many phenomena in physics, biology, climatology, economics, etc. which are often subject to uncertainty or random influences. Randomness may arise through stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions, for instance. One typical examples is the solution operator for a random differential equation driven by a real noise:

$$\frac{dx}{dt} = f(\theta_t\omega, x),$$

where $x \in \mathbb{R}^d$, $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function and $f_\omega(t, \cdot) \equiv f(\theta_t\omega, \cdot) \in L_{loc}(\mathbb{R}, C_b^{0,1})$. For details see [1], chapter 2.

Another example is the solution operator for a stochastic differential equation:

$$dx_t = f_0(x_t)dt + \sum_{k=1}^{d} f_k(x_t) \circ dB_t^k,$$

where $x \in \mathbb{R}^d$, $f_k, 0 \leq k \leq d$, are smooth vector fields, and $B_t = (B_t^1, \cdots, B_t^d)$ is the standard d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and dB_t^k is the Stratonovich differential. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the classic Wiener space, i.e., $\Omega = \{\omega : \omega(\cdot) \in C(\mathbb{R}, \mathbb{R}^d), \omega(0) = 0\}$ endowed with the open compact topology and $\mathbb{P}$ is the Wiener measure. Define a measurable dynamical system θ^t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by $(\theta^t\omega)(\cdot) = \omega(t+\cdot) - \omega(t)$ for $t > 0$. It is well-known that $\mathbb{P}$ is invariant and ergodic under θ^t. For more details about generators of random dynamical systems, see [1], chapter 2.

We consider a deterministic flow $\psi(t)(x) \equiv \psi(t, x)$ in $\mathbb{R}^n$ and its randomly perturbed flow (cocycle) $\phi(t, \omega)(x) \equiv \phi(t, \omega, x)$.

To state our main results, we first recall the persistence and existence results we obtained in [15]. We proved that for a C^r flow $\psi(t)$ with a compact, connected C^r normally hyperbolic invariant manifold $\mathcal{M} \subset \mathbb{R}^n$, there exists $\rho > 0$ such that for any C^r random flow $\phi(t, \omega)$ in $\mathbb{R}^n$ if

$$||\phi(t, \omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0, 1], \omega \in \Omega,$$

and the normal hyperbolicity is sufficiently strong, then $\phi(t, \omega)$ has a C^r normally hyperbolic random invariant manifold $\tilde{\mathcal{M}}(\omega)$ and has C^r stable manifold $\tilde{\mathcal{W}}^s(\omega)$ and unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ at $\tilde{\mathcal{M}}(\omega)$.

In this paper, with the above conditions and conclusions, we construct invariant foliations of the random stable and unstable manifolds $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ of $\tilde{\mathcal{M}}(\omega)$:

(i) *Stable Foliation: The stable manifold $\tilde{\mathcal{W}}^s(\omega)$ is foliated by an invariant family of stable leaves $\tilde{\mathcal{W}}^{ss}(\omega, x)$, i.e.,*

$$\tilde{\mathcal{W}}^s(\omega) = \cup_{x\in\tilde{\mathcal{M}}(\omega)}\tilde{\mathcal{W}}^{ss}(\omega, x).$$

(ii) *Unstable Foliation: The unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ is foliated by an invariant family of unstable leaves $\tilde{\mathcal{W}}^{uu}(\omega, x)$, i.e.,*

$$\tilde{\mathcal{W}}^u(\omega) = \cup_{x\in\tilde{\mathcal{M}}(\omega)}\tilde{\mathcal{W}}^{uu}(\omega, x).$$

The precise statement of our results is in the next section. When the perturbed flow $\phi(t, \omega, x)$ is a deterministic one, i.e., $\phi(t, \omega, x)$ is independent of ω, this result was proved by Fenichel [10]. It was also independently proved by Hirsch, Pugh and Shub [13].

Our results are applicable to systems of the form:

$$\frac{dx}{dt} = F(x) + \epsilon f(\theta_t\omega, x), \tag{1}$$

where f is C^1 and uniformly bounded in x, C^0 in t for fixed ω, and measurable in ω.

For stochastic differential equations of the form:

$$dx_t = F(x_t)dt + \epsilon f(x_t) \circ dB_t,$$

with the help of the Ornstein-Uhlenbeck process, we can sometimes transform the equation into a random differential equation of form (1), say:

$$\frac{dx}{dt} = \bar{F}(x) + \epsilon \bar{f}(\theta_t\omega, x).$$

However, the function $\bar{f}$ here is generally not uniformly bounded. So our results can not be applied directly. On the other hand, our results do provide some insight. If one truncates the Brownian motion by any fixed large constant, the conditions of our theorem are satisfied.

The theory of invariant foliations for deterministic dynamical systems has been well developed. The general theory for finite-dimensional dynamical systems near normally hyperbolic invariant manifolds was established by Fenichel [9, 10, 11] and Hirsch, Pugh, and Shub [13]. Pesin [21] proved a stable manifold theorem which gives an invariant foliation of the manifold. Ruelle [22] extended Pesin's result to semiflows with a compact invariant set in a Hilbert space. It was assumed that the linearized time-t map is compact and injective with dense range. The results are therefore applicable to some parabolic PDEs. Mañé [17] extended Pesin's result to semiflows in Banach space. A general local theory of invariant foliations for infinite-dimensional evolutionary equations was obtain by Chow, Lin, and Lu [7], while the global theory for infinite-dimensional deterministic dynamical systems was established in [2, 3, 4].

In [24], Wanner established the existence of invariant foliations for finite dimensional random dynamical systems in a neighborhood of a stationary solution and used the foliations to prove a Hartman-Grobman theorem for finite-dimensional RDSs. Li and Lu [19] proved a stable and unstable foliation theorem and used it to establish a smooth linearization theorem (Sternberg type of theorem) for finite-dimensional random dynamical systems. Pesin's result was established by Liu and

Qian [16] for finite-dimensional RDS and by Lian and Lu for infinite-dimensional RDS. The local theory of invariant foliations for stochastic PDEs was obtained by Lu and Schmalfuss [20]. The invariant foliations we present here are extensions of Fenichel's results to finite-dimensional random dynamical systems. It will be used in a forthcoming paper to establish a random Geometric Singular Perturbation Theory and to study slow-fast systems with random perturbations. Another main application of our result concerns the normal form near a compact slow manifold, which is a key step in proving a random version of the Inclination Lemma, also called the 'Exchange Lemma' [14]. We point out that there are two major difficulties: one is the lack of a deterministic Cartesian coordinate system in which the invariant foliation is constructed, the other is the measurability of the invariant foliation.

We organize this paper as follows: in Section 2 we state our main theorem; in Section 3 we prove the existence of the invariant foliation; in Section 4 we prove the smoothness and measurability of the invariant foliation; in Section 5 we prove the asymptotic property of the invariant foliation. Then in the last section, we discuss the extensions of our results to overflowing invariant and inflowing invariant manifolds.

2. Main result.

In this section, we first recall the concept of normally hyperbolic random invariant manifold and the persistence and existence results which are given in [15]. Then we state our main results.

Let $(\Omega, \mathcal{F}, P, \theta^t)$ be a metric dynamical system and let X be a separable Banach space. We consider a smooth random dynamical system $\phi(t, \omega, x)$ on X over θ^t.

A multifunction $\mathcal{M} = (\mathcal{M}(\omega))_{\omega \in \Omega}$ of nonempty closed sets $\mathcal{M}(\omega)$, $\omega \in \Omega$, contained in X is called a **random set** if

$$\omega \mapsto \inf_{y \in \mathcal{M}(\omega)} ||x - y||$$

is a random variable for any $x \in X$. When the random set $\mathcal{M}$ is a manifold, we call it a **random manifold**.

A random manifold $\mathcal{M}(\omega)$ is called a **random invariant manifold** for a random dynamical system $\phi(t, \omega, x)$ if

$$\phi(t, \omega, \mathcal{M}(\omega)) = \mathcal{M}(\theta^t \omega) \text{ for all } t \in \mathbb{R}, \omega \in \Omega.$$

Notation. By D we mean the derivative with respect to the spatial variable, while by D_1, D_2, and D_3 we mean the derivatives with respect to the first, second, and third variables, respectively. For instance, $D\phi(t, \omega)(x) \equiv D\phi(t, \omega, x) = D_x \phi(t, \omega, x)$, while $D_1 g(x, y, z) = D_x g$, $D_2 g(x, y, z) = D_y g$, and $D_3 g(x, y, z) = D_z g$. Moreover, for $\mathcal{M}$ a random manifold, by $(\omega, x) \in \Omega \times \mathcal{M}$ we mean $x \in \mathcal{M}(\omega)$.

Definition 2.1. A random invariant manifold $\mathcal{M}$ is said to be **normally hyperbolic** if for almost every $\omega \in \Omega$ and any $x \in \mathcal{M}(\omega)$, there exists a splitting which is C^0 in x and measurable:

$$X = E^u(\omega, x) \oplus E^c(\omega, x) \oplus E^s(\omega, x)$$

of closed subspaces with associated projections $\Pi^u(\omega, x)$, $\Pi^c(\omega, x)$, and $\Pi^s(\omega, x)$ such that

(i) The splitting is invariant:

$$D\phi(t, \omega)(x) E^i(\omega, x) = E^i(\theta_t \omega, \phi(t, \omega)(x)), \quad \text{for } i = u, c,$$

and

$$D\phi(t,\omega)(x)E^s(\omega,x) \subset E^s(\theta_t\omega, \phi(t,\omega)(x)).$$

(ii) $D\phi(t,\omega)(x)\big|_{E^i(\omega,x)} : E^i(\omega,x) \to E^i(\theta_t\omega, \phi(t,\omega)(x))$ is an isomorphism for $i = u, c$, and $E^c(\omega,x)$ is the tangent space of $\mathcal{M}(\omega)$ at x.

(iii) There are (θ,ϕ)-invariant random variables $\bar{\alpha}, \bar{\beta} : \mathcal{M} \to (0,\infty)$, $\bar{\alpha} < \bar{\beta}$, and a tempered random variable $K(\omega,x) : \mathcal{M} \to [1,\infty)$ such that

$$||D\phi(t,\omega)(x)\Pi_s(\omega,x)|| \le K(\omega,x)e^{-\bar{\beta}(\omega,x)t} \quad \text{for } t \ge 0, \tag{2}$$

$$||D\phi(t,\omega)(x)\Pi_u(\omega,x)|| \le K(\omega,x)e^{\bar{\beta}(\omega,x)t} \quad \text{for } t \le 0, \tag{3}$$

$$||D\phi(t,\omega)(x)\Pi_c(\omega,x)|| \le K(\omega,x)e^{\bar{\alpha}(\omega,x)|t|} \quad \text{for } -\infty < t < \infty. \tag{4}$$

The definition we have here is an extension of normal hyperbolicity for deterministic dynamical systems. Note that in the case of X being finite-dimensional,

$$D\phi(t,\omega)(x)\big|_{E^s(\omega,x)} : E^s(\omega,x) \to E^s(\theta_t\omega, \phi(t,\omega)(x))$$

is also an isomorphism.

We consider a deterministic flow $\psi(t)(x) \equiv \psi(t,x)$ in $\mathbb{R}^n$ and its randomly perturbed (cocycle) counterpart $\phi(t,\omega)(x) \equiv \phi(t,\omega,x)$. To state our main theorem, we recall the following theorem we obtained in [15].

Theorem 2.2. *Assume that $\psi(t)$ is a C^r flow, $r \ge 1$, which has a compact, connected C^r normally hyperbolic invariant manifold $\mathcal{M} \subset \mathbb{R}^n$. Let the positive exponents related to the normal hyperbolicity be $\bar{\alpha} < \bar{\beta}$ in (2.1)-(2.3), which in this case are constant and deterministic. Then there exists $\rho > 0$ such that for any random C^1 flow $\phi(t,\omega)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \textit{for all } t \in [0,1], \omega \in \Omega, \tag{5}$$

then

(i) Persistence: $\phi(t,\omega)$ has a C^1 normally hyperbolic random invariant manifold $\tilde{\mathcal{M}}(\omega)$ in a small neighborhood of $\mathcal{M}$,

(ii) Smoothness: If $\bar{\alpha} < r\bar{\beta}$ and $\phi(t,\omega)$ is C^r, then $\tilde{\mathcal{M}}(\omega)$ is a C^r manifold diffeomorphic to $\mathcal{M}$ for each $\omega \in \Omega$,

(iii) Existence of Stable Manifolds: $\tilde{\mathcal{M}}(\omega)$ has a stable manifold $\tilde{\mathcal{W}}^s(\omega)$ under $\phi(t,\omega)$,

(iv) Existence of Unstable Manifolds: $\tilde{\mathcal{M}}(\omega)$ has an unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ under $\phi(t,\omega)$.

Our first result is on the foliation of $\tilde{\mathcal{W}}^u(\omega)$ into unstable fibers based on $\tilde{\mathcal{M}}(\omega)$.

Theorem 2.3. *Assume the conditions of Theorem 2.2 hold. Then there exists a unique C^{r-1} in x family of C^r submanifolds $\{\tilde{\mathcal{W}}^{uu}(\omega,x) : \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ of $\tilde{\mathcal{W}}^u(\omega)$ satisfying:*

(1) For each $(\omega,x) \in \Omega \times \tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{uu}(\omega,x) = \{x\}$, $T_x\tilde{\mathcal{W}}^{uu}(\omega,x) = E^u(\omega,x)$ and $\tilde{\mathcal{W}}^{uu}(\omega,x)$ varies measurably with respect to (ω,x) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If x_1, $x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \ne x_2$, then $\tilde{\mathcal{W}}^{uu}(\omega,x_1) \cap \tilde{\mathcal{W}}^{uu}(\omega,x_2) = \emptyset$ and

$$\tilde{\mathcal{W}}^u(\omega) = \cup_{x\in\tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega,x).$$

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t,\omega)\big(\tilde{\mathcal{W}}^{uu}(\omega,x)\big) \subset \tilde{\mathcal{W}}^{uu}(\theta_t\omega, \phi(t,\omega)x)$.

(4) *For* $y \in \tilde{\mathcal{W}}^{uu}(\omega, x)$ *and* $x \neq x_1 \in \tilde{\mathcal{M}}(\omega)$ *with* $|\phi(t,\omega)(x_1) - \phi(t,\omega)(x)| \to 0$ *as* $t \to -\infty$, *we have*

$$\frac{|\phi(t,\omega)(y) - \phi(t,\omega)(x)|}{|\phi(t,\omega)(y) - \phi(t,\omega)(x_1)|} \to 0$$

exponentially as $t \to -\infty$.

(5) *For* $y_1, y_2 \in \tilde{\mathcal{W}}^{uu}(\omega, x)$, $|\phi(t,\omega)(y_1) - \phi(t,\omega)(y_2)| \to 0$ *exponentially as* $t \to -\infty$.

(6) $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ *is* C^0 *in* t *for any fixed* (ω, x).

The next result is on the foliation of $\tilde{\mathcal{W}}^s(\omega)$ into stable fibers based on $\tilde{\mathcal{M}}(\omega)$.

Theorem 2.4. *Assume the conditions of Theorem 2.2 hold. Then, there exists a unique* C^{r-1} *in* x *family of* C^r *submanifolds* $\{\tilde{\mathcal{W}}^{ss}(\omega, x) : \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ *of* $\tilde{\mathcal{W}}^s(\omega)$ *satisfying:*

(1) *For each* $(\omega, x) \in \Omega \times \tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{ss}(\omega, x) = \{x\}$, $T_x\tilde{\mathcal{W}}^{ss}(\omega, x) = E^s(\omega, x)$ *and* $\tilde{\mathcal{W}}^{ss}(\omega, x)$ *varies measurably with respect to* (ω, x) *in* $\Omega \times \tilde{\mathcal{M}}$.

(2) *If* $x_1, x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \neq x_2$, *then* $\tilde{\mathcal{W}}^{ss}(\omega, x_1) \cap \tilde{\mathcal{W}}^{ss}(\omega, x_2) = \emptyset$ *and*

$$\tilde{\mathcal{W}}^s(\omega) = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{ss}(\omega, x).$$

(3) *For* $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t,\omega)\big(\tilde{\mathcal{W}}^{ss}(\omega, x)\big) \subset \tilde{\mathcal{W}}^{ss}(\theta_t\omega, \phi(t,\omega)x)$.

(4) *For* $y \in \tilde{\mathcal{W}}^{ss}(\omega, x)$ *and* $x \neq x_1 \in \tilde{\mathcal{M}}(\omega)$ *with* $|\phi(t,\omega)(x_1) - \phi(t,\omega)(x)| \to 0$ *as* $t \to \infty$, *we have*

$$\frac{|\phi(t,\omega)(y) - \phi(t,\omega)(x)|}{|\phi(t,\omega)(y) - \phi(t,\omega)(x_1)|} \to 0$$

exponentially as $t \to +\infty$.

(5) *For* $y_1, y_2 \in \tilde{\mathcal{W}}^{ss}(\omega, x)$, $|\phi(t,\omega)(y_1) - \phi(t,\omega)(y_2)| \to 0$ *exponentially as* $t \to +\infty$.

(6) $\tilde{\mathcal{W}}^{ss}(\theta^t\omega, x)$ *is* C^0 *in* t *for any fixed* (ω, x).

To prove the theorems, some useful properties proved in [15] will be used. Let

$$\mathbb{R}^n = E^u(\omega, x) \oplus E^c(\omega, x) \oplus E^s(\omega, x)$$

be the splitting corresponding to the normal hyperbolicity of $\tilde{\mathcal{M}}(\omega)$.

Proposition 1. *For* ρ *sufficiently small, the following holds*

(i) *The splitting is invariant:*

$$D\phi(t,\omega)(x)E^i(\omega, x) = E^i(\theta_t\omega, \phi(t,\omega)(x)), \quad \text{for } i = u, c, s,$$

$D\phi(t,\omega)(x)\big|_{E^i(\omega,x)} : E^i(\omega, x) \to E^i(\theta_t\omega, \phi(t,\omega)(x))$ *is an isomorphism for* $i = u, c, s$. $E^c(\omega, x)$ *is the tangent space of* $\mathcal{M}(\omega)$ *at* x. $E^i(\omega, x)$ *are measurable in* (ω, x) *and* C^{r-1} *in* x.

(ii) (1) *There exist positive constants* $a < 1$ *and* c_1 *such that:*

$$||\tilde{\Pi}^s D\phi(t, \theta^{-t}\omega)\phi(-t,\omega)(x)|_{\tilde{E}^s(\theta^{-t}\omega)}|| < c_1 a^t$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ *and* $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t, \theta^{-t}\omega)\phi(-t,\omega)(x)|_{\tilde{E}^u(\theta^{-t}\omega)}|| < c_1 a^t$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ *and* $t \leq 0$, *(2) If $\bar{\alpha} < r\bar{\beta}$, there exist $c_2 > 0$ and $r' > r$ such that*

$$||\tilde{\Pi}^s D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(x)|_{\tilde{E}^s(\theta^{-t}\omega)}|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\omega)})(-t, \omega)(x)||^{r'} < c_2$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(x)|_{\tilde{E}^u(\theta^{-t}\omega)}|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\omega)})(-t, \omega)(x)||^{r'} < c_2$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$,

(3) If $\bar{\alpha} < r\bar{\beta}$, there exist $c_3 > 0$ and $r' > r$ such that

$$||\tilde{\Pi}^s D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(x)|_{\tilde{E}^s(\theta^{-t}\omega)}|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\omega)})(-t, \omega)(x)|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\theta^{-t}\omega)})(t, \theta^{-t}\omega)\phi(-t, \omega)(x)||^{r'-1} < c_3$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(x)|_{\tilde{E}^u(\theta^{-t}\omega)}|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\omega)})(-t, \omega)(x)|| \, ||D(\phi|_{\tilde{\mathcal{M}}(\theta^{-t}\omega)})(t, \theta^{-t}\omega)\phi(-t, \omega)(x)||^{r'-1} < c_3$$

for all $x \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

3. Existence of the invariant foliation.

In this section, we prove the existence of an invariant foliation of the random unstable manifold. We construct the invariant foliation in local coordinates on $\tilde{\mathcal{W}}^u(\omega)$. The basic idea is due to Hadamard [12] and involves a graph transform. First, we take a set of Lipschitz graphs in local charts, which pass through different points on the random invariant manifold $\tilde{\mathcal{M}}(\omega)$ and are contained in the random unstable manifold $\tilde{\mathcal{W}}^u(\omega)$. Then, we consider the image of these graphs under $\phi(t, \omega)$ for some large fixed $t > 0$. We show that this random graph transform is a contraction on the space of such sets of graphs and the resulting fixed set of graphs gives us the invariant foliation of the random unstable manifold. By reversing time, we get an invariant foliation of the random stable manifold. The technical hurdle is the construction of local charts. We need the local charts on $\tilde{\mathcal{W}}^u(\omega)$ for different $\omega \in \Omega$, while at the same time we need those charts to be related to each other for different ω. We overcome this difficulty by using the fact that the random unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ and deterministic unstable manifold $\mathcal{W}^u$ are C^r diffeomorphic and C^1-close. Thus, we can define local coordinates on $\mathcal{W}^u$ and then induce local coordinates on $\tilde{\mathcal{W}}^u(\omega)$. We denote by $i(\omega)$ the C^r diffeomorphism from $\mathcal{W}^u$ to $\tilde{\mathcal{W}}^u(\omega)$. For any fixed ω, $i(\omega)$ is C^r close to the identity map Id.

Notation. We have used x and y for points on manifolds in $\mathbb{R}^n$. From now on, we will use m for points on $\mathcal{M}$ or $\tilde{\mathcal{M}}(\omega)$ and use x with superscript u, s, cu, cs for normal coordinates. It will be made clear from the context. In this and those sections after, there will be all kinds of random bundles. For any random bundle $E = E(\omega, m)$, we use $E(\epsilon_i)$ to mean the subset $\{\nu \in E(\omega, m)$ for some $(\omega, m), |\nu| < \epsilon_i\}$.

To define local coordinates on $\mathcal{W}^u$, we follow Fenichel's approach, see [10]. Let exp be the exponential map. For each $m \in \mathcal{M}$ and $\nu \in T\mathcal{W}^u|\mathcal{M}$, let $exp_m(\nu)$ be the end point of the geodesic with initial point m and initial tangent vector ν. We borrow the following lemma from [10]:

Lemma 3.1. *There exists $0 < \epsilon_1$ such that for each $m \in \mathcal{M}$,*

$$exp_m : \{\nu \in T_m\mathcal{W}^u|_{\mathcal{M}} : |\nu| < \epsilon_1\} \to \mathcal{W}^u$$

is a diffeomorphism onto its range and its range lies in $\mathcal{W}^u$. Moreover, the derivative of the exponential map $D\,exp_m(0)$ is the identity map and $||D\,exp_m(\nu)||$ and $||[D\,exp_m(\nu)]^{-1}||$ are arbitrarily close to 1 uniformly for $m \in \mathcal{M}$ and $|\nu| < \epsilon_1$ if ϵ_1 is small enough.

Fix ϵ_1 small enough such that the conclusions of Lemma 3.1 hold. Define

$$\Gamma(\omega) : T\tilde{\mathcal{W}}^u(\omega)|_{\tilde{\mathcal{M}}(\omega)} \mapsto \tilde{\mathcal{W}}^u(\omega)$$

by

$$\Gamma(\omega) := i(\omega) \circ exp \circ [D\, i(\omega)]^{-1}.$$

This gives us local coordinates on $\tilde{\mathcal{W}}^u(\omega)$ near $\tilde{\mathcal{M}}(\omega)$. Taking sufficiently small ρ, by the uniform C^1-closeness of $\psi(t)$ to $\phi(t,\omega)$, $\mathcal{W}^u$ to $\tilde{\mathcal{W}}^u(\omega)$ (see Theorem 2.2 for what we mean by uniform), and Lemma 3.1, there exists $0 < \epsilon_2 < \epsilon_1$ such that $\Gamma(\omega)$ is well defined on $(T\tilde{\mathcal{W}}^u(\omega)|_{\tilde{\mathcal{M}}(\omega)})(\epsilon_2)$.

By Proposition 1, if K is sufficiently large, we have

$$||D\phi(-K,\omega)(m)|_{E^u(\omega,m)}|| < \frac{1}{4},$$

$$||D((\phi|_{\tilde{\mathcal{M}}(\theta^{-K}\omega)})(K,\theta^{-K}\omega))\phi(-K,\omega)(m)||^k ||D\phi(-K,\omega)(m)|_{E^u(\omega,m)}|| < \frac{1}{4},$$

for any $0 \le k \le r$. (Note that the first inequality is in a different form, which does not matter because Proposition 1 holds for arbitrary ω. Also note that for fixed K and sufficiently small ρ, $\phi(\pm K,\omega)$ is uniformly C^1 close to $\psi(\pm K)$ by (5) and the cocycle property.) Fixing this K, there exists $0 < \epsilon_3 < \epsilon_2$ such that

$$\phi(K,\omega)\{\Gamma(\omega)((T\tilde{\mathcal{W}}^u(\omega)|_{\tilde{\mathcal{M}}(\omega)})(\epsilon_3))\} \subset \Gamma(\theta^K\omega)((T\tilde{\mathcal{W}}^u(\theta^K\omega)|_{\tilde{\mathcal{M}}(\theta^K\omega)})(\epsilon_2)).$$

For any $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, let $\xi^c,\ \xi^u,\ x^c,\ x^u$ denote elements of

$$T\tilde{\mathcal{M}}(\theta^{-K}\omega, \phi(-K,\omega)(m)), E^u(\theta^{-K}\omega, \phi(-K,\omega)(m)), T\tilde{\mathcal{M}}(\omega,m) \text{ and } E^u(\omega,m)$$

respectively. We use (ξ^c,ξ^u) and (x^c,x^u) as coordinates near $\phi(-K,\omega)(m)$ and m, respectively. The map $\phi(K,\theta^{-K}\omega)$ has the form

$$(\xi^c,\xi^u) \ \mapsto \ (x^c,x^u) \ = \ (g^c(\xi^c,\xi^u), g^u(\xi^c,\xi^u)),$$

defined for $|\xi| := |\xi^c| + |\xi^u| < \epsilon_3$. In terms of g^c and g^u, the above inequalities read

$$||[D_2 g^u(0,0)]^{-1}|| = ||D\phi(-K,\omega)(m)|_{E^u(\omega,m)}|| < \frac{1}{4},$$

$$||[D_2 g^u(0,0)]^{-1}|| ||D_1 g^c(0,0)||^k < \frac{1}{4}, \text{ for } 0 \le k \le r.$$

We also have from the invariance of $\tilde{\mathcal{M}}(\omega)$ and $E^i(\omega),\ i = u,c$, the following

$$g^u(0,0) = 0, \quad g^c(0,0) = 0, \quad D_2 g^c(0,0) = 0.$$

Note that g^c and g^u depend on ξ^c, ξ^u as well as on m and ω. And there exists Q so large that all first partial derivatives of g^c and g^u along with their inverses are bounded by Q.

By the compactness of $\mathcal{M}$, the C^1-closeness of $\psi(K)$ to $\phi(K,\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ to $\mathcal{W}^u$, uniformly in ω, for any constants $\beta > 0$ and $\gamma > 0$ there exists $0 < \epsilon_4 < \epsilon_3$ such that for all $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, if $|\xi^c|,\ |\bar{\xi^c}|,\ |\xi^u|,\ |\bar{\xi^u}| \le \epsilon_4$, then

$$||[D_2 g^u(\xi^c,\xi^u)]^{-1}|| < \frac{1}{3}, \tag{6}$$

$$||[D_2g^u(\xi^c,\xi^u)]^{-1}||||D_1g^c(\bar{\xi}^c,\bar{\xi}^u)||^k < \frac{1}{3},\ \text{for } 0 \le k \le r, \tag{7}$$

$$||g^u(\xi^c,\xi^u)|| < \gamma,\quad ||g^c(\xi^c,\xi^u)|| < \gamma,\quad ||D_2g^c(\xi^c,\xi^u)|| < \gamma, \tag{8}$$

$$|g^u(\xi^c,\xi^u) - g^u(\xi^c,\bar{\xi}^u)| \ge [||[D_2g^u(\xi^c,\xi^u)]^{-1}||^{-1} - \beta]\,|\xi^u - \bar{\xi}^u|, \tag{9}$$

$$|g^c(\xi^c,\xi^u) - g^c(\bar{\xi}^c,\xi^u)| \le [||D_1g^c(\xi^c,\xi^u)|| + \beta]\,|\xi^c - \bar{\xi}^c|. \tag{10}$$

For β sufficiently small, there exists a small positive δ_0 such that if

$$|\xi^c - \bar{\xi}^c| \le \delta_0|\xi^u - \bar{\xi}^u|,$$

then

$$\begin{aligned} |g^u(\xi^c,\xi^u) - g^u(\bar{\xi}^c,\bar{\xi}^u)| \ge & |g^u(\xi^c,\xi^u) - g^u(\xi^c,\bar{\xi}^u)| - |g^u(\xi^c,\bar{\xi}^u) - g^u(\bar{\xi}^c,\bar{\xi}^u)| \\ \ge & [||[D_2g^u(\xi^c,\xi^u)]^{-1}||^{-1} - \beta]\,|\xi^u - \bar{\xi}^u| - Q|\xi^c - \bar{\xi}^c| \\ \ge & (3 - \beta - Q\delta_0)\,|\xi^u - \bar{\xi}^u| > 2|\xi^u - \bar{\xi}^u|. \end{aligned} \tag{11}$$

Let S denote the set of families of continuous maps

$$h = \{h(\omega,m) : \omega \in \Omega,\ m \in \tilde{\mathcal{M}}(\omega), h(\omega,m)(0) = 0\},$$

where $h(\omega,m)(\cdot) :\ E^u(\omega,m)(\epsilon_4) \to T\tilde{\mathcal{M}}(\omega,m)(\epsilon_4)$ is also continuous in the base point m.

For $h \in S$, define

$$Lip\, h := \sup_{\omega\in\Omega} \max_{m\in\tilde{\mathcal{M}}(\omega)} \sup_{x^u,\bar{x}^u\in E^u(\omega,m)(\epsilon_4), x^u\neq\bar{x}^u} \frac{|h(\omega,m)(x^u) - h(\omega,m)(\bar{x}^u)|}{|x^u - \bar{x}^u|},$$

if it exists. Define

$$S_\delta := \{h \in S : Lip\, h \le \delta\},$$

and a distance d on S_δ:

$$\begin{aligned} d(h,h') := \quad & \sup\{\frac{|h(m,\omega)(x^u) - h'(m,\omega)(x^u)|}{|x^u|}\ :\ \omega \in \Omega, \\ & m \in \tilde{\mathcal{M}}(\omega),\ 0 \neq x^u \in E^u(m,\omega)(\epsilon_4)\}. \end{aligned}$$

Note that set S_δ is nonempty since the trivial family is obviously in S_δ and that the above supremum exists because each term is bounded by 2δ. Under this metric, S_δ is complete. Moreover, convergence in S_δ implies uniform convergence.

We will construct an $h \in S_\delta$ such that $\tilde{\mathcal{W}}^{uu}(\omega,m)$ is the graph of $h(\omega,m)$.

Proposition 2. *There exists a unique point in S_δ, which we denote by h, such that for any $t > K$, h satisfies the overflowing invariance condition:*

$$\phi(-t,\omega)(graph(h(\omega,m))) \subset graph(h(\theta^{-t}\omega,\phi(-t,\omega)(m))).$$

Proof. We first note that in local coordinates the above overflowing invariance condition is equivalent to the nonlinear functional equation:

$$h(\omega,m)(g^u(h(\theta^{-K}\omega,\phi(-K,\omega)(m))(\xi^u)),\xi^u) = g^c(h(\theta^{-K}\omega,\phi(-K,\omega)(m))(\xi^u)),\xi^u).$$

We will show that this functional equation has a unique solution in S_δ.

Define a map G on $S_\delta \to S$ as follows. For $h \in S_\delta,\ \omega \in \Omega,\ m \in \tilde{\mathcal{M}}(\omega)$,

$$(Gh)(m,\omega)(x^u)\ =\ g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u),$$

where

$$\begin{aligned} x^u\ &=\ g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u), \\ m'\ &=\ \phi(-K,\omega)(m). \end{aligned}$$

The next lemma justifies this definition.

Lemma 3.2. *If δ and ϵ_4 are sufficiently small, for each $h \in S_\delta$, $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, the map $\xi^u \mapsto g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)$ is one-to-one on $E^u(\theta^{-K}\omega, m')(\epsilon_4)$ and $E^u(\omega, m)(\epsilon_4)$ is contained in its range.*

Proof. By (11) we conclude that g^u is one-to-one. Then g^u is a continuous injection from an open subset of Euclidean space to Euclidean space of the same dimension. By invariance of domain, the range of g^u is open. Since $g^u(0,0) = 0$, there exists $c > 0$ such that $B(0,c)$ is contained in the range of g^u. Again from (11) we have that the pre-image of $B(0,c)$ is contained in $B(0,c/2)$ and that $B(0,\epsilon_4)$ is contained in the range of g^u. □

Lemma 3.3. *If δ and ϵ_4 are sufficiently small, G maps S_δ into S_δ.*

Proof. It is obvious that $(Gh)(\omega, m)(0) = 0$. We only need to estimate the Lipschitz constant of Gh. Let $x^u,\ \bar{x}^u \in E^u(\omega, m)(\epsilon)$ and define $\xi^u,\ \bar{\xi}^u$ by

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u),$$

$$\bar{x}^u = g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u),$$

which are well defined by lemma 3.2. Then

$$\begin{aligned} |\bar{x}^u - x^u| &= |g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\ &\geq |g^u(h(\theta^{-K}\omega, m')(\xi^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\ &\quad -|g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \bar{\xi}^u)| \\ &\geq \{||[D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta\}|\bar{\xi}^u - \xi^u| \\ &\quad -Q|h(\theta^{-K}\omega, m')(\bar{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\ &\geq \{||[D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta\}|\bar{\xi}^u - \xi^u|. \end{aligned}$$

Also, we have

$$\begin{aligned} & |(Gh)(\omega, m)(\bar{x}^u) - (Gh)(\omega, m)(x^u)| \\ = \ & |g^c(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\ \leq \ & [||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]|h(\theta^{-K}\omega, m')(\bar{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\ & +\gamma|\bar{\xi}^u - \xi^u| \\ \leq \ & \{[||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]\delta + \gamma\}|\bar{\xi}^u - \xi^u|. \end{aligned}$$

So by (6) and (7), for δ small enough, choosing ϵ_4 and γ sufficiently small, we have *Lip* $Gh < \delta$. (Note that γ could be arbitrarily small by taking sufficiently small ϵ_4.) □

Lemma 3.4. *If δ and ϵ_4 are sufficiently small, G is a contraction on S_δ.*

Proof. Let $h,\ \hat{h} \in S_\delta$ and $x^u \in E^u(\omega, m)(\epsilon_4)$. Then, there exist

$$\xi^u,\ \hat{\xi}^u \in E^u(\theta^{-K}\omega, m')(\epsilon_4)$$

such that $x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) = g^u(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u)$. By (11), we have

$$\begin{aligned}
2|\xi^u - \hat{\xi^u}| &\leq |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \hat{\xi}^u)| \\
&= |g^u(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \hat{\xi}^u)| \\
&\leq Q|\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\
&\leq Q|\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u) - \hat{h}(\theta^{-K}\omega, m')(\xi^u)| \\
&\quad +Q|\hat{h}(\theta^{-K}\omega, m')(\xi^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\
&\leq Q\delta|\hat{\xi^u} - \xi^u| + Qd(\hat{h}, h)|\xi^u|.
\end{aligned}$$

Choosing δ such that $\delta < \frac{1}{Q}$, we have

$$|\xi^u - \hat{\xi^u}| \leq Qd(\hat{h}, h)|\xi^u|.$$

We also have

$$\begin{aligned}
&|(Gh)(\omega, m)(x^u) - (G\hat{h})(\omega, m)(x^u)| \\
= \; &|g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^c(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u)| \\
\leq \; &|g^c(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u) - g^c(\hat{h}(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
&+|g^c(\hat{h}(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
\leq \; &[(Q+\beta)\delta + \gamma]|\xi^u - \hat{\xi^u}| + [||D_1g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]d(h, \hat{h})|\xi^u| \\
\leq \; &(Q+\beta+\gamma/\delta)\delta\; Qd(h, \hat{h})|\xi^u| + [||D_1g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]d(h, \hat{h})|\xi^u| \\
= \; &[||D_1g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta + (Q+\beta+\gamma/\delta)\delta\; Q]d(h, \hat{h})|\xi^u|
\end{aligned}$$

and

$$\begin{aligned}
|x^u| &= |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
&= |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^u(0,0)| \\
&\geq [||[D_2g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta]|\xi^u|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{|(Gh)(\omega, m)(x^u) - (G\hat{h})(\omega, m)(x^u)|}{|x^u|} \\
\leq \; &\frac{||D_1g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta + (Q+\beta+\gamma/\delta)\delta\; Q}{||D_2g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta} d(h, \hat{h}).
\end{aligned}$$

Choosing δ and ϵ sufficiently small and using (7) the factor preceding $d(h, \hat{h})$ can be bounded by a constant $\lambda < 1$. Thus, we have

$$d(Gh, G\hat{h}) \leq \lambda d(h, \hat{h}).$$

This completes the proof of the lemma. □

By the contraction principle, there exists a unique fixed point h of G in S_δ. h satisfies:

$$\phi(-K, \omega)(\text{graph}(h(\omega, m))) \subset \text{graph}(h(\theta^{-K}\omega, \phi(-K, \omega)(m))).$$

For any fixed $t > K$ we can define G_1 just as we defined G for K. We know that G and G_1 commute. So we have

$$GG_1h = G_1Gh = G_1h.$$

By the uniqueness of G we conclude

$$G_1 h = h.$$

Or equivalently

$$\phi(-t,\omega)(\mathrm{graph}(h(\omega,m))) \subset \mathrm{graph}(h(\theta^{-t}\omega, \phi(-t,\omega)(m))).$$

This completes the proof of Proposition 2. □

4. **Smoothness and measurability.**

In this section, we prove two types of smoothness of the invariant foliation: the smoothness of each fiber and the smoothness with respect to the base point. We also prove that the fiber changes measurably with respect to ω and the base point jointly.

The strategy for proving smoothness is the same as in [15], which is the same as in [9, 10, 11]. We first differentiate the equation of the fixed point formally to determine the functional equation, which must be satisfied by the derivatives. Second, we show the functional equation has a unique solution in some space. Last, we show that this unique solution is indeed the derivative. The idea of proving measurability of the invariant foliation is to prove it is the limit of a sequence of measurable foliations.

Notation. We have used D for spatial derivatives. In this section, there may be two kinds of spatial variables, $h(\omega,m)(x^u)$ for instance. $D^k h(\omega,m)(x^u)$ will mean the k-th order derivative with respect to x^u. We have also used D_i for the derivative of the i-th variable. In this section, with the explicit dependence on ω of functions, $h(\omega)(x^{cc},x^u)$ for instance, $D_1 h(\omega)(x^{cc},x^u)$ will mean the derivative with respect to x^{cc}.

We split this section into three subsections:

4.1. **Smoothness of the fiber.**

By proving the following proposition, we show that each fiber of the invariant foliation is C^r.

Proposition 3. *For any $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, $h(m,\omega)(x^u)$ is a C^r function of x^u and all derivatives $D^k h(m,\omega)(x^u), 1 \le k \le r$ are continuous in the base point m.*

Proof. First, we have

$$h(\omega,m)(g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)) = g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u). \tag{12}$$

Taking the derivative formally on both sides of the above equation, we have

$$\begin{aligned} & Dh(\omega,m)(x^u)[D_1 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u) \\ & +D_2 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)] \\ = \ & D_1 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u) + D_2 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u). \end{aligned}$$

Thus, if $h(\omega, m)$ is differentiable, then we must have

$$\begin{aligned} & Dh(\omega, m)(x^u) \\ = \; & [D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) Dh(\theta^{-K}\omega, m')(\xi^u) \\ & + D_2 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)] \cdot \\ & [D_1 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) Dh(\theta^{-K}\omega, m')(\xi^u) \\ & + D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}, \end{aligned}$$

where

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u), \text{ and } m' = \phi(-K, \omega)(m).$$

The candidate for Dh, which we denote by v, has the following form:

$$v = \{v(m, \omega) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\},$$

where for each $\omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)$, $v(m, \omega)(\cdot)$ is a continuous map from

$$E^u(m, \omega)(\epsilon_4) \to L(E^u(m, \omega), T\tilde{\mathcal{M}}(\omega, m)),$$

or equivalently,

$$v(m, \omega) \in C^0(E^u(m, \omega)(\epsilon_4), L(E^u(m, \omega), T\tilde{\mathcal{M}}(\omega, m))).$$

Let TS be the space of all such v. Define the norm $||\cdot||$ on TS by

$$||v|| = \sup_{\omega, m} \max_{x^u \in E^u(m, \omega)} ||v(m, \omega)(x^u)||.$$

Under this norm, TS is complete. We want to find an element $v \in TS$ such that

$$\begin{aligned} & v(\omega, m)(x^u) \\ = & [D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) v(\theta^{-K}\omega, m')(\xi^u \\ & + D_2 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)] \cdot \\ & [D_1 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) v(\theta^{-K}\omega, m')(\xi^u) \\ & + D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}, \end{aligned} \tag{13}$$

where

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u), \text{ and } m' = \phi(-K, \omega)(m).$$

We prove the functional equation (13) for v has a unique solution in TS.

Define a sequence $\{v^n\} \subset TS$ by induction: Let $v^0 \equiv 0$ and

$$\begin{aligned} & v^{n+1}(\omega, m)(x^u) \\ = & [D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) v^n(\theta^{-K}\omega, m')(\xi^u) \\ & + D_2 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)] \\ & [D_1 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) v^n(\theta^{-K}\omega, m')(\xi^u) \\ & + D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}, \end{aligned} \tag{14}$$

where

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u), \text{ and } m' = \phi(-K, \omega)(m).$$

We have the following two lemmas, the proofs of which follow exactly the same strategy as in Section 5 of [15].

Lemma 4.1. $||v^n|| < \delta$ *for all* n.

Lemma 4.2. $||v^{n+1} - v^n|| \le \lambda ||v^n - v^{n-1}||$ *for some* $0 < \lambda < 1$.

Hence, v^n converge to the unique solution v of equation (13). By the uniform convergence, $v(m,\omega)(x^u)$ is continuous in x^u and m.

The next lemma states that Proposition 3 holds for the case $k = 1$.

Lemma 4.3. $Dh(m,\omega)(x^u) = v(m,\omega)(x^u)$.

Proof. For a fixed $\omega \in \Omega$, we define an increasing function $\varrho_\omega : (0,1) \to \mathbb{R}$ by,

$$\begin{aligned} &\varrho_\omega(a) \\ =\ & \max_{m \in \tilde{\mathcal{M}}(\omega)} \sup_{x^u, x^{u\prime} \in E^u(m,\omega), 0<||x^u - x^{u\prime}||<a} \\ & \frac{||h(m,\omega)(x^{u\prime}) - h(m,\omega)(x^u) - v(m,\omega)(x^u)(x^{u\prime} - x^u)||}{||x^{u\prime} - x^u||}. \end{aligned}$$

Note that ϱ_ω is bounded by 2δ.

We want to show $\varrho_\omega(a) \to 0$ as $a \to 0$. To prove this, we claim

Claim. $\varrho_\omega(a)$ *satisfies*

$$\varrho_\omega(a) \le \alpha \varrho_{\omega'}(\kappa a) + r(\omega', a)$$

for small a, *where* $0 \le \alpha < 1$, $1 < \kappa$, $\omega' = \theta^{-K}\omega$, *and* $r(\theta^{-K}\omega, a)$ *is a decreasing function in* a, *approaching zero as* $a \to 0$ *uniformly with respect to* $\omega \in \Omega$.

The proof of the claim is exactly the same as in Proposition 5.2 of [15].

Replace successively a by $a\kappa^{-1}, a\kappa^{-2}, \cdots, a\kappa^{-n}$ and ω by $\theta^K\omega, \theta^{2K}\omega, \cdots, \theta^{nK}\omega$, respectively, weight the terms with $1, \frac{1}{\alpha}, \cdots, \frac{1}{\alpha^{n-1}}$, and add them together to get:

$$\frac{1}{\alpha^{n-1}} \varrho_{\theta^{nK}\omega}(a\kappa^{-n}) \le \alpha \varrho_\omega(a) + (1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{n-1}}) \sup_\omega \max_{0 \le t \le a} r(\omega, t).$$

Then since $\varrho_\omega(a) \le 2\delta$, we have

$$\varrho_{\theta^{nK}\omega}(a\kappa^{-n}) \le 2\alpha^n \delta + \frac{1}{1-\alpha} \sup_\omega \max_{0 \le t \le a} r(\omega, t).$$

Since ω is arbitrary, we get

$$\varrho_\omega(a\kappa^{-n}) \le 2\alpha^n \delta + \frac{1}{1-\alpha} \sup_\omega \max_{0 \le t \le a} r(\omega, t).$$

It follows that $\varrho_\omega(a) \to 0$ as $a \to 0$.

So $h(m,\omega)(x^u)$ is C^1 in x^u and $Dh(m,\omega)(x^u)$ is continuous in the base point m. □

Lemma 4.4. $D^{k-1}v(m,\omega)(x^u)$ *exists for* $2 \le k \le r$ *and is continuous in* x^u *and* m. *Moreover* $D^k h(m,\omega)(x^u) = D^{k-1}v(m,\omega)(x^u)$.

Proof. By induction, it is easy to see $D^k v^n$ is a Cauchy sequence in the corresponding space, which implies the uniform convergence of $D^k v^n(m,\omega)$. Since v^n converges to v, we have that $D^k v(m,\omega)(x^u)$ exists and equals the uniform limit of $D^k v^n(m,\omega)(x^u)$ □

Combining Lemma 4.3 and Lemma 4.4 gives Proposition 3. □

Proposition 4. *The graph of* $h(\omega, m)$ *is tangent to* $E^u(\omega, m)$ *at* m.

Proof. It is equivalent to show $Dh(\omega,m)(x^u)|_{x^u=0}=0$.

Since
$$|h(\omega,m)(x^u)-h(\omega,m)(\bar{x}^u)|\le\delta|x^u-\bar{x}^u|,$$
we have
$$|Dh(\omega,m)(x^u)|\le\delta$$
for all $|x^u|<\epsilon_4$.

Recall that, in proving that the graph transform is a contraction in Section 3, for each small δ, we let ϵ_4 be small enough to ensure the existence. If we let $\delta\to 0$, it follows that $\epsilon_4\to 0$. So by uniqueness of the fibers, the above inequality always holds for $x^u=0$. Equivalently,
$$|Dh(\omega,m)(0)|\le\delta$$
for arbitrary $\delta>0$. It follows that $Dh(\omega,m)(0)=0$. □

4.2. **Smoothness with respect to the base point.** In this subsection, we prove the fiber changes C^{r-1} smoothly as the base point varies on the center manifold. We will introduce a new coordinate system and use the same idea as in the last subsection.

Proposition 5. *$h(\omega,m)$ is C^{r-1} in m for $m\in\tilde{\mathcal{M}}$.*

Proof. We need to prove the random C^0 manifold $\Sigma=\{\Sigma(\omega):\omega\in\Omega\}$ defined by
$$\Sigma(\omega)=\{(m,p)|m\in\tilde{\mathcal{M}}(\omega),p\in\tilde{\mathcal{W}}^{uu}(\omega,m)\}$$
is a C^{r-1} submanifold of $\tilde{\mathcal{M}}\times\tilde{\mathcal{W}}^u=\{\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega):\omega\in\Omega\}$.

Let $\tilde{\mathcal{M}}_*=\{\tilde{\mathcal{M}}_*(\omega):\omega\in\Omega\}$ be the diagonal embedding of $\tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}}\times\tilde{\mathcal{W}}^u$:
$$\tilde{\mathcal{M}}_*(\omega):=\{(m,m)|m\in\tilde{\mathcal{M}}(\omega)\}.$$
Then, $\tilde{\mathcal{M}}_*$ is a compact connected C^r random invariant manifold in $\mathbb{R}^n\times\mathbb{R}^n$ under (ϕ,ϕ).

We embed $T\tilde{\mathcal{W}}^u(\omega)$ into $T(\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega))|_{\tilde{\mathcal{M}}_*(\omega)}$ as follows. Let $\gamma(t)$ be a curve in $\tilde{\mathcal{W}}^u(\omega)$ such that $\gamma(0)=m\in\tilde{\mathcal{M}}(\omega)$. Then $\gamma^*(t):=(m,\gamma(t))$ is a curve in $\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega)$ such that $\gamma^*(0)=(m,m)$. The mapping $\gamma\to\gamma^*$ induces an injection $T\tilde{\mathcal{W}}^u(\omega)|_{\tilde{\mathcal{M}}(\omega)}\to T(\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega))|_{\tilde{\mathcal{M}}_*(\omega)}$.

Let $E^u_*(\omega)$ and $E^c_*(\omega)$ be the image of $E^u(\omega)$ and $T\tilde{\mathcal{M}}(\omega)$ under this injection, respectively. Actually, only vectors whose norms are smaller than ϵ_4 are embedded. However, this is not important. We will consider only vectors with small norms. Then, we get the following splitting:
$$T(\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega))|_{\tilde{\mathcal{M}}_*(\omega)}=T\tilde{\mathcal{M}}_*(\omega)\oplus E^u_*(\omega)\oplus E^c_*(\omega).$$
The embedding we have here is based on [11].

To prove Proposition 5, we need local coordinates and partitions of unity along the lines of those used in [15]. We first present the proof in the case that $\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega)$ is a subset of a torus and $E^u_*(\omega)$ and $E^c_*(\omega)$ are trivial bundles. In other words, we have a global coordinate system. Later, we explain how this proof is to be modified to fit the general case.

Denote the global coordinates by $(x^{cc},x^u,x^c)\in T\tilde{\mathcal{M}}_*(\omega)\times E^u_*(\omega)(\epsilon_5)\times E^c_*(\omega)(\epsilon_5)$. We may choose small ϵ_5 as we did for ϵ_3, such that the induced random flow $\phi^*(K,\omega):=(\phi(K,\omega),\phi(K,\omega))$ on $\tilde{\mathcal{M}}(\omega)\times\tilde{\mathcal{W}}^u(\omega)$ has the form:
$$j(x^{cc},x^u,x^c)=(j^{cc}(x^{cc},x^u,x^c),j^u(x^{cc},x^u,x^c),j^c(x^{cc},x^u,x^c)).\qquad(15)$$

From Proposition 1 we have

$$||[D_2 j^u(x^{cc},0,0)]^{-1}|| < \frac{1}{4}, \tag{16}$$

$$||[D_2 j^u(x^{cc},0,0)]^{-1}||||D_3 j^c(x^{cc},0,0)||^k < \frac{1}{4}, \tag{17}$$

$$||[D_1 j^{cc}(x^{cc},0,0)]^{-1}||^{k-1}||[D_2 j^u(x^{cc},0,0)]^{-1}||||D_3 j^c(x^{cc},0,0)|| < \frac{1}{4} \tag{18}$$

for $1 \le k \le r$. We can choose ϵ_5 even smaller as for the case of ϵ_4, such that

$$||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}|| < \frac{1}{3}, \tag{19}$$

$$||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}||||D_3 j^c(x^{cc},x^u,x^c)||^k < \frac{1}{3}, \tag{20}$$

$$||[D_1 j^{cc}(x^{cc},x^u,x^c)]^{-1}||^{k-1}||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}||||D_3 j^c(x^{cc},x^u,x^c)|| < \frac{1}{3}. \tag{21}$$

From the invariance of $\tilde{\mathcal{M}}_*(\omega)$, we have

$$j^u(x^{cc},0,0) = 0, \;\; j^c(x^{cc},0,0) = 0, \tag{22}$$

and so

$$D_1 j^u(x^{cc},0,0) = 0, \;\; D_1 j^c(x^{cc},0,0) = 0. \tag{23}$$

By the invariance of $E^u_*(\omega)$ we get

$$D_2 j^c(x^{cc},0,0) = 0 \tag{24}$$

and

$$D_1 D_2 j^c(x^{cc},0,0) = 0. \tag{25}$$

Hence, for any small γ, choosing ϵ_5 small enough we have

$$|j^u(x^{cc},x^u,x^c)| < \gamma, \;\; |j^c(x^{cc},x^u,x^c)| < \gamma, \tag{26}$$

$$||D_1 j^u(x^{cc},x^u,x^c)|| < \gamma, \;\; ||D_1 j^c(x^{cc},x^u,x^c)|| < \gamma, \tag{27}$$

$$||D_2 j^c(x^{cc},x^u,x^c)|| < \gamma, \;\; ||D_1 D_2 j^c(x^{cc},x^u,x^c)|| < \gamma. \tag{28}$$

Moreover, we may suppose all first and second partial derivatives of j^{cc}, j^u, j^c are bounded by some $Q > 0$. Let this Q be large enough that it is an upper bound of all bounded terms which may come later.

We represent $\Sigma(\omega)$ by

$$h^*(\omega) : T\tilde{\mathcal{M}}_*(\omega,m^*) \times E^u_*(\omega,m^*) \to E^c_*(\omega,m^*).$$

From Proposition 3, $h^*(\omega)(x^{cc},x^u)$ is C^r in x^u and $D_2^k h^*(\omega)(x^{cc},x^u)$ is C^0 in x^{cc} for $0 \le k \le r$, and the following hold

$$h^*(\omega)(x^{cc},0) = 0, \; D_2 h^*(\omega)(x^{cc},0) = 0, \; ||D_2 h^*(\omega)(x^{cc},x^u)|| \le \delta,$$

$$||D_2 h^*(\omega)(x^{cc},x^u)|| = ||D_2 h^*(\omega)(x^{cc},x^u) - D_2 h^*(\omega)(x^{cc},0)|| \le Q(\omega)|x^u|.$$

By the uniform C^r closeness of all $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ to each other, the C^1 closeness of $\phi(K,\omega)$ to the deterministic $\psi(K)$, uniformly in ω, and the compactness, we get a uniform estimate

$$||D_2 h^*(\omega)(x^{cc},x^u)|| \le Q|x^u|. \tag{29}$$

From the invariance of $\Sigma(\omega)$ we obtain

$$h^*(\omega)(x^{cc},x^u) = j^c(\xi^{cc},\xi^u,h^*(\theta^{-K}\omega)(\xi^{cc},\xi^u)), \tag{30}$$

where

$$x^{cc} = j^{cc}(\xi^{cc},\xi^u,h^*(\theta^{-K}\omega)(\xi^{cc},\xi^u)),$$

$$x^u = j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)).$$

Taking the derivative with respect to ξ^{cc} formally on both side of (30) gives

$$\begin{aligned} &D_1 h^*(\omega)[D_1 j^{cc} + D_3 j^{cc} D_1 h^*(\theta^{-K}\omega)] \\ &+ D_2 h^*(\omega)[D_1 j^u + D_3 j^u D_1 h^*(\theta^{-K}\omega)] \\ =&D_1 j^c + D_3 j^c D_1 h^*(\theta^{-K}\omega), \end{aligned} \tag{31}$$

where the arguments of h^*, j are clear from the context.

For any fixed $\omega \in \Omega$, let $v^*(\omega) \in C^0(T\tilde{\mathcal{M}}_*(\omega) \times E_*^u(\omega), L(T\tilde{\mathcal{M}}_*(\omega), E_*^c(\omega)))$ and $v^* = \{v^*(\omega) : \omega \in \Omega\}$. Define

$$||v^*||_{Lip} = \sup_{\omega} \sup_{x^u \neq 0} \frac{||v^*(\omega)(x^{cc}, x^u)||}{|x^u|}. \tag{32}$$

Let DS denote the space of all such v^* with norm given by (32). Under this norm, DS is a complete metric space. We prove the following functional equation of $v^* \in DS$ has a unique solution:

$$\begin{aligned} &v^*(\omega)(x^{cc}, x^u)[D_1 j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) \\ &\qquad + D_3 j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ &+ D_2 h^*(\omega)(x^{cc}, x^u)[D_1 j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) \\ &\qquad + D_3 j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ =&D_1 j^c(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) \\ &\qquad + D_3 j^c(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u), \end{aligned} \tag{33}$$

where

$$x^{cc} = j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)),$$

$$x^u = j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)).$$

We follow the approach of Lemma 4.3.

Define a sequence $\{v_*^n\} \subset DS$ by

$$\begin{aligned} v_*^0 &= 0, \\ v_*^{n+1}(\omega) &= [D_1 j^c + D_3 j^c D_1 h^*(\theta^{-K}\omega) - D_2 h^*(\omega)(D_1 j^u + D_3 j^u v_*^n(\theta^{-K}\omega))] \\ &\qquad [D_1 j^{cc} + D_3 j^{cc} v_*^n(\theta^{-K}\omega)]^{-1}, \end{aligned}$$

where the arguments of $h^*(\omega)$ and v_*^{n+1} are (x^{cc}, x^u), the arguments of $h^*(\theta^{-K}\omega)$ and v_*^n are (ξ^{cc}, ξ^u) and the arguments of $D_1 j^{cc}$, $D_3 j^{cc}$, $D_1 j^u$, $D_3 j^u$, $D_1 j^c$, $D_3 j^c$ are $(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{xx}, \xi^u))$.

We prove $\{v_*^n\}$ is a Cauchy sequence in DS.

Lemma 4.5. $||v_*^{n+1}||_{Lip} \leq \delta$.

Proof. The proof of this lemma is straightforward following from (21),(26),(27),(28) and (29). $\square$

Lemma 4.6. $||v_*^{n+1} - v_*^n||_{Lip} < \lambda ||v_*^n - v_*^{n-1}||_{Lip}$ *for some* $0 < \lambda < 1$.

Proof. First, we note that

$$\begin{aligned} &v_*^{n+1}(\omega)(x^{cc},x^u)[D_1j^{cc}+D_3j^{cc}v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)] \\ &\quad + D_2h^*(\omega)[D_1j^u+D_3j^uv_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)] \\ &= D_1j^c+D_3j^cv_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u), \end{aligned} \tag{34}$$

$$\begin{aligned} &v_*^{n}(\omega)(x^{cc},x^u)[D_1j^{cc}+D_3j^{cc}v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)] \\ &\quad + D_2h^*(\omega)[D_1j^u+D_3j^uv_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)] \\ &= D_1j^c+D_3j^cv_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u), \end{aligned} \tag{35}$$

where the arguments of D_1j^{cc}, D_3j^{cc}, D_1j^u, D_3j^u, D_1j^c, D_3j^c are

$$(\xi^{cc},\xi^u,h^*(\theta^{-K}\omega)(\xi^{xx},\xi^u)).$$

From (34)−(35), we get

$$\begin{aligned} &v_*^{n+1}(\omega)(x^{cc},x^u)-v_*^n(\omega)(x^{cc},x^u) \\ =&[D_3j^c-v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)D_3j^{cc}-D_2h^*(\omega)(x^{cc},x^u)D_3j^u] \\ &\cdot[v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)-v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)] \\ &\cdot[D_1j^{cc}+D_3j^{cc}v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)]^{-1}. \end{aligned} \tag{36}$$

We also have

$$\begin{aligned} |x^u| &= |j^u(\xi^{cc},\xi^u,h(\theta^{-K}\omega)(\xi^{cc},\xi^u))| \\ &= |j^u(\xi^{cc},\xi^u,h(\theta^{-K}\omega)(\xi^{cc},\xi^u))-j^u(\xi^{cc},0,h(\theta^{-K}\omega)(\xi^{cc},0))| \\ &\geq (||(D_2j^u)^{-1}||^{-1}-\beta)|\xi^u|-Q\delta|\xi^u|. \end{aligned} \tag{37}$$

From (21),(36) and (37), it is easy to get

$$\begin{aligned} &\frac{||v_*^{n+1}(\omega)(x^{cc},x^u)-v_*^n(\omega)(x^{cc},x^u)||}{|x^u|} \\ <&\frac{1}{2}\frac{||v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)-v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)||}{|\xi^u|}, \end{aligned}$$

which gives us

$$||v_*^{n+1}-v_*^n||_{\text{Lip}} \leq \frac{1}{2}||v_*^n-v_*^{n-1}||_{\text{Lip}}$$

□

Let v^* be the uniform limit of $\{v_*^n\}$. We prove that v^* is the partial derivative D_1h^* of h^*. Along the lines of Lemma 4.3, we see that v^* is the partial derivative D_1h^* of h^*. We already have that D_2h^* exists and is C^0. So h^* is C^1 jointly in (m,x^u).

The rest of the proof of Proposition 5 is straightforward, proceeding along the lines of Proposition 3. □

Remark. $h(\omega,m,x^u)$ is actually C^{r-1} jointly in (m,x^u).

Besides all the above smoothness properties, the invariant foliation has the following continuity property:

Proposition 6. *$\tilde{\mathcal{W}}^{uu}(\theta^t\omega,x)$ is C^0 in t for any fixed (ω,x).*

Proof. Suppose $m(0)$ is a point on the fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$ represented by $x^u(0) + h(\omega, m, x^u(0))$ in local coordinates. From the invariance of the foliation, we have $\phi(t, \omega, m(0)) \in \tilde{\mathcal{W}}^{uu}(\theta^t\omega, \phi(t, \omega, m))$. So $\phi(t, \omega, m(0))$ can be represented in local coordinates by

$$x^u(t) + h(\theta^t\omega, \phi(t, \omega, m), x^u(t)).$$

Since $\phi(t, \omega, m(0))$ is C^0 in t, $x^u(t) + h(\theta^t\omega, \phi(t, \omega, m), x^u(t))$ is C^0 in t. Then since $x^u(t)$ and $\phi(t, \omega, m)$ are both C^0 in t, it must follow that $h(\theta^t\omega, m, x^u)$ is C^0 in t, which gives the conclusion of the proposition. ☐

Remark. From the above proof, we conclude that the smoothness of the fibers $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ in t for any fixed (ω, x) is the same as the smoothness of any orbit of the random system.

For the general case where no global chart exists, we construct local charts on $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}^u$ near $\tilde{\mathcal{M}}_*$ using a similar method to that in Section 3. Let $\mathcal{M} \times \mathcal{W}^u$ and $\mathcal{M}_*$ be the deterministic counterparts of $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}^u$ and $\tilde{\mathcal{M}}_*$, respectively. A result similar to Lemma 3.1 holds in this setting. In other words, Lemma 3.1 holds if we replace $\mathcal{M}$ and $\mathcal{W}^u$ there by $\mathcal{M}_*$ and $\mathcal{M} \times \mathcal{W}^u$ respectively. The compactness of $\mathcal{M}_*$ gives us local charts on $\mathcal{M} \times \mathcal{W}^u$ near $\mathcal{M}_*$. The C^r closeness of $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ to $\mathcal{M} \times \mathcal{W}^u$, uniformly in ω induces local charts on $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}^u$ near $\tilde{\mathcal{M}}_*$ in exactly the same way as in Section 3, after Lemma 3.1.

In the local charts, the induced random flow $\phi^*(K, \omega)$ has exactly the same form as (15):

$$j(x^{cc}, x^u, x^c) = (j^{cc}(x^{cc}, x^u, x^c), j^u(x^{cc}, x^u, x^c), j^c(x^{cc}, x^u, x^c)),$$

with the understanding that j depends on $m^* \in \tilde{\mathcal{M}}_*(\omega)$ as well. All the estimates are uniform in m and ω. So the proof also holds in the case of local charts.

4.3. **Measurability of the fibers.** In this subsection, we prove that the fibers in the unique family $\{\tilde{\mathcal{W}}^{uu}(\omega, m) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}$ change in a measurable way with respect to ω.

What we need to do is to prove the representation $h^*(\omega, x^{cc}, x^u)$ is measurable. The major difficulty is that the coordinate system we used to construct the unique family depends on ω. In other words, the coordinates x^{cc} and x^u of $h^*(\omega, x^{cc}, x^u)$ depend on ω. To overcome this problem, we use the measurability and smoothness of $\tilde{\mathcal{M}}_*(\omega)$, $E^u_*(\omega)$ and $E^c_*(\omega)$ to construct ω-independent coordinates in $\mathbb{R}^m \oplus \mathbb{R}^l \oplus \mathbb{R}^m$.

Lemma 4.7. *There exists a coordinate system in which h^* has a new form*

$$\tilde{h}^*(\omega, y^{cc}, y^u)$$

with the following properties: y^{cc} and y^u are independent of ω; $\tilde{h}^*(\omega, y^{cc}, y^u)$ is C^r in y^u and C^{r-1} in (y^{cc}, y^u) jointly.

Proof. Fix any $\omega_0 \in \Omega$, let $m_0(\omega_0) \in \tilde{\mathcal{M}}_*(\omega_0)$. Since for different $\omega \in \Omega$, all $\tilde{\mathcal{M}}_*(\omega)$ are C^r diffeomorphic to each other, we get a set of points $m_0(\omega) \in \tilde{\mathcal{M}}_*(\omega)$ corresponding to $m_0(\omega_0) \in \tilde{\mathcal{M}}_*(\omega_0)$. Then by the measurable selection theorem [6], there exist measurable bases

$$\{\bar{e}^u_1(\omega, m_0(\omega)), \cdots, \bar{e}^u_l(\omega, m_0(\omega))\} \text{ and } \{\bar{e}^c_1(\omega, m_0(\omega)), \cdots, \bar{e}^c_m(\omega, m_0(\omega))\},$$

of the tangent spaces $E^u_*(\omega, m_0(\omega))$ and $E^c_*(\omega, m_0(\omega))$.

From the C^{r-1} smoothness of $\tilde{\mathcal{M}}_*(\omega_0)$, $E^u_*(\omega_0)$ and $E^c_*(\omega_0)$, there exist bases of the bundles $E^u_*(\omega_0)$ and $E^c_*(\omega_0)$:

$$\{e^u_1(\omega_0, m(\omega_0)), \cdots, e^u_l(\omega_0, m(\omega_0))\} \text{ and } \{e^c_1(\omega_0, m(\omega_0)), \cdots, e^c_m(\omega_0, m(\omega_0))\},$$

which are C^{r-1} in $m(\omega_0)$ and satisfy:

$$e^i_k(\omega_0, m_0(\omega_0)) = \bar{e}^i_k(\omega_0, m_0(\omega_0)),$$

for $i = u$, $k = 1, \cdots, l$ and $i = c$, $k = 1, \cdots, m$.

Note that in the above $m(\omega_0) \in \tilde{\mathcal{M}}_*(\omega_0)$ is not a function of ω but can vary in $\tilde{\mathcal{M}}_*(\omega_0)$.

For each fixed $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}_*(\omega)$, by the same reasoning, we get possibly different C^{r-1} bases of the bundles $E^u_*(\omega)$ and $E^c_*(\omega)$:

$$\{e^u_1(\omega, m), \cdots, e^u_l(\omega, m)\} \text{ and } \{e^c_1(\omega, m), \cdots, e^c_m(\omega, m)\},$$

Claim. *These bases can be chosen jointly measurable in* (ω, m).

Notice that for fixed m_0, they are measurable in ω. From the C^{r-1} smoothness, the measurability of the bundle and the C^{r-1} diffeomorphism of the bundles to each other and to the deterministic counterpart, it follows that they are measurable in ω and C^{r-1} in m and the claim follows.

Therefore, there exist a neighborhood D of 0 in $\mathbb{R}^m \oplus \mathbb{R}^l \oplus \mathbb{R}^m$ and a map

$$T(\omega, \cdot) : \tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega) \to D$$

such that $T(\omega, \cdot)$ is a C^{r-1} diffeomorphism for each ω and $T(\cdot, z)$, $T^{-1}(\cdot, z)$ are measurable for each $z \in \tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$. Here, $T^{-1}(\cdot, z)$ refers to the family of maps parameterized by ω which are, for fixed z, the inverses of $T(\omega, \cdot)$ at $T(\omega, z)$. Moreover, points on $\tilde{\mathcal{M}}_*(\omega) = \{(m, m) | m \in \tilde{\mathcal{M}}\}$ are mapped to $D \cap (\mathbb{R}^m \times \{0\} \times \{0\})$ and $e^u_i(\omega, m)$ are mapped to unit vectors in the e_i directions in $\mathbb{R}^l$ and $e^c_j(\omega, m)$ are mapped to unit vectors in the e_j directions in $\mathbb{R}^m$.

h^* has the form $\tilde{h}^*(\omega, y^{cc}, y^u)$ in this new coordinate system D. Obviously, all the properties listed in the lemma are satisfied by $\tilde{h}^*$. □

Our next step is to prove that $\tilde{h}^*(\omega, y^{cc}, y^u)$ is measurable. The following lemma from [5] will be used:

Lemma 4.8. *Given a polish space* H *and any mapping*

$$P : \Omega \times H \to H,$$

satisfying that $P(\omega, \cdot)$ *is a homeomorphism for any* $\omega \in \Omega$, *and* $P(\cdot, x), P^{-1}(\cdot, x)$ *are measurable for any* $x \in H$, *if* ϕ *is a continuous random dynamical system, then so is* ϕ' *defined by*

$$\phi'(t, \omega, x) := P(\theta^t \omega, \phi(t, \omega, P^{-1}(\omega, x))).$$

Recall that, $\phi^*(t, \omega, x)$ is a C^r random flow in the original ω-dependent coordinate system. Under the new ω-independent coordinate system, $\phi^*(t, \omega, x)$ has the form

$$\tilde{\phi}^*(t, \omega, y) = T(\theta^t \omega, \phi^*(t, \omega, T^{-1}(\omega) y)).$$

By the above lemma, $\tilde{\phi}^*(t, \omega, y)$ is a C^{r-1} random flow.

Proposition 7. $\tilde{h}^*(\omega, y^{cc}, y^u)$ *is* C^{r-1} *in* (y^{cc}, y^u) *and measurable in* ω, *so is measurable in* (ω, y^{cc}, y^u).

Proof. Recall that, in Section 3, we constructed the invariant foliation by finding the unique fixed point of a contraction mapping G on S_δ (graph transform). So the invariant foliation is the limit of any starting foliations (starting element of S_δ) under iterations of the mapping G.

Suppose $h_0(\omega, x, m)$ is the representation of a starting foliation. After one iteration under the graph transform, $h_0(\omega, x, m)$ becomes $h_1(\omega, x, m)$. By Proposition 2, we have the relationship between $h_0(\omega, x, m)$ and $h_1(\omega, x, m)$: for $h_0 \in S_\delta,\ \omega \in \Omega,\ m \in \tilde{\mathcal{M}}(\omega)$,

$$h_1(m, \omega)(x^u) \;=\; g^c(h_0(\theta^{-K}\omega, m')(\xi^u), \xi^u)$$

where

$$\begin{aligned} x^u &= g^u(h_0(\theta^{-K}\omega, m')(\xi^u), \xi^u), \\ m' &= \phi(-K, \omega)(m). \end{aligned}$$

From the above we see that as long as $\phi(t, \omega, x)$ is measurable in ω and C^r in x, g^u and g^c are measurable in ω and C^r in other coordinates. Therefore, h_1 has the same measurability and smoothness properties as h_0.

Now, we consider it in the new coordinate system. We note that $\tilde{h}^*(\omega, y^{cc}, y^u)$ is the limit of a sequence $\tilde{h}_n^*(\omega, y^{cc}, y^u)$ which is generated by iterating the graph of $\tilde{h}_0^*(\omega, y^{cc}, y^u)$ under the graph transform G^*, where G^* is generated by the random flow $\tilde{\phi}^*(t, \omega, y)$.

Because $\tilde{\phi}^*(t, \omega, y)$ is a C^{r-1} random flow, i.e., measurable in ω and C^{r-1} in y, as long as we take $\tilde{h}_0^*(\omega, y^{cc}, y^u) \equiv 0$, which is C^{r-1} in (y^{cc}, y^u) and measurable in ω, we get C^{r-1} smoothness and ω-measurability of all the sequence $\tilde{h}_n^*(\omega, y^{cc}, y^u)$. Therefore, the limit $\tilde{h}^*(\omega, y^{cc}, y^u)$ is also measurable in ω.

On the other hand, since the change of the coordinate system is given by $T(\omega, \cdot)$ which is a C^{r-1} diffeomorphism for each ω, $\tilde{h}^*(\omega, y^{cc}, y^u)$ is C^{r-1}. Therefore, $\tilde{h}^*(\omega, y^{cc}, y^u)$ is measurable in (ω, y^{cc}, y^u). ☐

Summing up the results of this section, we get the following

Proposition 8. *The unique family of fibers $\{\tilde{\mathcal{W}}^{uu}(\omega, m) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}$ is a C^{r-1} in m family of C^r manifolds and the fibers in it change measurably with ω. Moreover, $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ is C^0 in t for any fixed (ω, x).*

5. **Asymptotic property.** In this section, we prove that the points on an unstable fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$ are equivalent in a certain asymptotic sense and characterize the invariant foliation. The technical difficulty is the lack of a uniform metric (distance) on the random unstable (stable) manifold. To overcome this, we again use the C^r diffeomorphism and C^r closeness of the random unstable manifold to the corresponding deterministic one.

Since we will not use the smoothness in the base point nor the measurability of the invariant foliation in this section, we will use the coordinate system used in Section 3.

Suppose $\hat{d}(\omega)(\cdot, \cdot)$ is the geodesic distance on $\tilde{\mathcal{W}}^u(\omega)$: For any $m, m' \in \tilde{\mathcal{W}}^u(\omega)$, $\hat{d}(m, m')$ is the infimum of the lengths of piecewise smooth rectifiable curves in $\tilde{\mathcal{W}}^u(\omega)$ joining m and m', if any such curve exists. Otherwise $\hat{d}(m, m') = \infty$. Let $d(\cdot, \cdot)$ be the geodesic distance on $\mathcal{W}^u$. Then d induces a distance $\tilde{d}(\omega)$ on $\tilde{\mathcal{W}}^u(\omega)$ in the following manner:

$$\tilde{d}(\omega)(m, m') := \inf\{\text{length of } c(t)\}$$

for $c(t), t \in [0,a]$ a piecewise smooth rectifiable curve joining $i^{-1}(\omega, m)$ and $i^{-1}(\omega, m')$ in $\mathcal{W}^u$. Since $\mathcal{W}^u$ and $\tilde{\mathcal{W}}^u(\omega)$ are C^1 close, and $i(\omega)$ are C^1 close to Id, uniformly in ω, we conclude that $\tilde{d}(\omega)$ is uniformly equivalent to $\hat{d}(\omega)$, the geodesic distance on $\tilde{\mathcal{W}}^u(\omega)$. Under $\tilde{d}$, we have

$$\tilde{d}(\omega)(\Gamma(\omega, m, \nu), m) = |Di^{-1}(\omega, m)\nu|.$$

Recall that Γ was used to define local coordinates in Section 3. (See the discussion after Lemma 3.1.) Since $Di(\omega, m)$ and $Di(\omega, m)^{-1}$ are uniformly close to the identity matrix transformation, we can define another uniformly equivalent distance $d(\omega)$ on $\tilde{\mathcal{W}}^u(\omega)$ such that

$$d(\omega)(\Gamma(\omega, m, \nu), m) = |\nu|.$$

We will use $d(\omega)$ to measure the distance on $\tilde{\mathcal{W}}^u(\omega)$. To simplify notation, we use d in place of $d(\omega)$. We have the following proposition which characterizes the fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$.

Proposition 9. *Suppose $m, m' \in \tilde{\mathcal{M}}(\omega)$, $p \in \tilde{\mathcal{W}}^{uu}(\omega, m)$ and $p' \in \tilde{\mathcal{W}}^{uu}(\omega, m')$, then*

(i) $d(\phi(-t,\omega)(p), \phi(-t,\omega)(m)) \to 0$ exponentially as $t \to \infty$;
(ii) If $m \neq m'$ and $d(\phi(-t,\omega)(m), \phi(-t,\omega)(m')) \to 0$ as $t \to \infty$, then

$$\frac{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m))}{d(\phi(-t,\omega)(p'), \phi(-t,\omega)(m))} \to 0 \text{ as } t \to \infty,$$

$$\frac{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m))}{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m'))} \to 0 \text{ as } t \to \infty;$$

(iii) $\tilde{\mathcal{W}}^{uu}(\omega, m) \cap \tilde{\mathcal{W}}^{uu}(\omega, m') = \emptyset$ if $m \neq m'$;
(iv) $\tilde{\mathcal{W}}^u(\omega) = \cup_{m \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega, m)$.

Proof. From Proposition 1 we have for $K > 0$ as in Section 3

$$||D\phi(-K,\omega)(m)|_{E^u(\omega)}|| < \frac{1}{4}a_1^K,$$

and

$$||D((\phi|_{\tilde{\mathcal{M}}(\theta^{-K}\omega)})(K, \theta^{-K}\omega))\phi(-K,\omega)(m)||^r ||D\phi(-K,\omega)(m)|_{E^u(\omega)}|| < \frac{1}{4},$$

which yield that for some $a_1 < a_2 < 1$,

$$||D\phi(-K,\omega)(m)|_{E^u(\omega)}|| < \frac{1}{4}a_1^K < \frac{1}{4}a_2^K,$$

$$||D((\phi|_{\tilde{\mathcal{M}}(\theta^{-K}\omega)})(K, \theta^{-K}\omega))\phi(-K,\omega)(m)||^k ||D\phi(-K,\omega)(m)|_{E^u(\omega)}|| < \frac{1}{4}a_2^K,$$

where k is no larger than r.

As we obtained the estimates (6), (7) and (11), we get similar estimates:

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| < \frac{1}{3}a_1^K, \tag{38}$$

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| ||D_1 g^c(\bar{\xi}^c, \bar{\xi}^u)||^k < \frac{1}{3}a_2^K, \tag{39}$$

and

$$|g^u(h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u)| > 2\frac{|\xi^u|}{a_1^K}, \tag{40}$$

where $m_1 = \phi(-K,\omega)(m)$. We have

$$d(\Gamma(\omega, m, \nu), m) = |\nu|,$$

for $\omega \in \Omega$, $m \in \tilde{\mathcal{M}}(\omega)$ and $|\nu| < \epsilon_4$. If δ is sufficiently small, and $x^u \in E^u(\omega, m)(\epsilon_4)$, $x^c \in T\tilde{\mathcal{M}}(\omega, m)(\epsilon_4)$ such that $|x^c| \le \delta |x^u|$, for all $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, then

$$\frac{3}{4}|x^u| \le d(\Gamma(\omega, m, (x^u, x^c)), m) \le \frac{4}{3}|x^u|. \tag{41}$$

Moreover, without the condition that $|x^c| \le \delta |x^u|$, there is a constant c_5 such that

$$d(\Gamma(\omega, m, (x^u, x^c)), m) \ge c_5 |x^c|.$$

This is because the angle between $E^u(\omega, m)$ and $T\tilde{\mathcal{M}}(\omega, m)$ is bounded away from zero uniformly.

To prove part (i) and (ii), it is enough to let t approach infinity through multiples of K.

(i) Let $p = \Gamma(\omega, m, (h(\omega, m)(x^u), x^u))$,

$$\phi(-K, \omega, p) = \Gamma(\theta^{-K}\omega, m_1, (h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u)).$$

Then we have $x^u = g^u(h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u)$ and by (40), (41),

$$\begin{aligned} d(\phi(-K,\omega)(p), \phi(-K,\omega)(m)) &\le \frac{4}{3}|\xi^u| \le \frac{4}{3}\cdot\frac{1}{2}a_1^K g^u(h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u) \\ &= \frac{2}{3}a_1^K |x^u| \le \frac{8}{9}a_1^K d(p, m), \end{aligned}$$

which leads to the conclusion of part (i).

(ii) There exists N large enough such that for $n \ge N$,

$$d(\phi(-nK,\omega)(m), \phi(-nK,\omega)(m'))$$

are so small that $\phi(-nK,\omega)(m')$ can be represented in local coordinates near $m_n := \phi(-nK,\omega)(m)$ as $(\hat{\xi}_n^c, \hat{\xi}_n^u)$, while $\phi(-nK,\omega)(p)$ is represented as (ξ_n^c, ξ_n^u) where $\xi_1^c = \xi^c$, $\xi_1^u = \xi^u$, $\hat{\xi}_1^c = \hat{\xi}^c$, $\hat{\xi}_1^u = \hat{\xi}^u$ and $\xi_0^c = x^c$, $\xi_0^u = x^u$ $\hat{\xi}_0^c = \hat{x}^c$, $\hat{\xi}_0^u = \hat{x}^u$ as before.

Without loss of generality, we may assume that for any $n \ge 0$, $\phi(-nK,\omega)(m')$ can be represented in local coordinates near m_n.

Since $m' \in \tilde{\mathcal{M}}(\omega)$ and $\tilde{\mathcal{M}}(\omega)$ is invariant under $\phi(-t,\omega)$, we have

$$|\hat{\xi}_n^c| \;>\; \delta|\hat{\xi}_n^u|$$

for any $n \ge 0$. In particular,

$$|\hat{\xi}^c| \;>\; \delta|\hat{\xi}^u|.$$

Now

$$\begin{aligned} |x^u| &= |g^u(h(\theta^{-K}\omega, m_1)(\xi^u), (\xi^u))| \\ &\ge |g^u(h(\theta^{-K}\omega, m_1)(\xi^u), (\xi^u)) - g^u(h(\theta^{-K}\omega, m_1)(\xi^u), 0)| \\ &\quad -|g^u(h(\theta^{-K}\omega, m_1)(\xi^u), 0) - g^u(0,0)| \\ &\ge [|||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta]|\xi^u| - Q\delta|\xi^u| \end{aligned}$$

and

$$\begin{aligned} |\hat{x}^c| &= |g^c(\hat{\xi}^c, \hat{\xi}^u)| \\ &\le |g^c(\hat{\xi}^c, \hat{\xi}^u) - g^c(0, \hat{\xi}^u)| + |g^c(0, \hat{\xi}^u) - g^c(0,0)| \\ &\le [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta]|\hat{\xi}^c| + \gamma|\hat{\xi}^u| \\ &\le [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \frac{\gamma}{\delta}]|\hat{\xi}^c| \\ &= [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \tau]|\hat{\xi}^c|. \end{aligned}$$

So

$$\frac{|\xi^u|}{|\hat{\xi}^c|} \le \frac{|x^u|}{|\hat{x}^c|} \, \frac{||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \tau}{||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta - Q\delta} \le \frac{|x^u|}{|\hat{x}^c|} a_2^K.$$

Similarly

$$\frac{|\xi_n^u|}{|\hat{\xi}_n^c|} \le \frac{|\xi_{n-1}^u|}{|\hat{\xi}_{n-1}^c|} a_2^K \le \cdots \le \frac{|x^u|}{|\hat{x}^c|} a_2^{nK}.$$

Since

$$d(m', m) = |(\hat{x}^c, \hat{x}^u)| = \sqrt{|\hat{x}^c|^2 + |\hat{x}^u|^2} \le \sqrt{\frac{1}{\delta^2} + 1}\, |\hat{x}^c| < (1 + \frac{1}{\delta})|\hat{x}^c|,$$

we conclude

$$\frac{d(\phi(-nK, \omega, p), \phi(-nK, \omega, m))}{d(\phi(-nK, \omega, m'), \phi(-nK, \omega, m))} \le \frac{\frac{3}{4}|\xi_n^u|}{c_5|\hat{\xi}_n^c|} \le \frac{3}{4c_5} a_2^{nK} \frac{|x^u|}{|\hat{x}^c|} \to 0 \text{ as } n \to \infty.$$

This gives us

$$\frac{d(\phi(-t, \omega)(p), \phi(-t, \omega)(m))}{d(\phi(-t, \omega)(m'), \phi(-t, \omega)(m))} \to 0 \text{ exponentially as } t \to \infty. \tag{42}$$

Part (ii) is a consequence of (42).

(iii) Suppose $q \in \tilde{\mathcal{W}}^{uu}(\omega, m) \cap \tilde{\mathcal{W}}^{uu}(\omega, m')$ for $m \ne m'$. By part (i),

$$d(\phi(-t, \omega)(q), \phi(-t, \omega)(m)) \to 0 \text{ and } d(\phi(-t, \omega)(q), \phi(-t, \omega)(m')) \to 0$$

as $t \to \infty$. Hence $d(\phi(-t, \omega)(m'), \phi(-t, \omega)(m)) \to 0$. Then by part (ii),

$$1 \le \frac{d(\phi(-t, \omega)(q), \phi(-t, \omega)(m)) + d(\phi(-t, \omega)(q), \phi(-t, \omega)(m'))}{d(\phi(-t, \omega)(m'), \phi(-t, \omega)(m))} \to 0,$$

which is a contradiction.

(iv) Suppose (U, Φ) is a local chart on $\tilde{\mathcal{M}}(\omega)$ about a point $m \in \tilde{\mathcal{M}}(\omega)$ such that $E^u(\omega)$ has a C^{r-1} orthonormal basis in U. Let (V, Ψ) be a local chart on $\tilde{\mathcal{W}}^u(\omega)$ about m. Define a map $\chi : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^{m+l}$ by

$$\chi(x^c, x^u) = \Psi(\Gamma(\omega, \Phi^{-1}x^c, (h(\omega, \Phi^{-1}x^c)(x^u), x^u))).$$

Then this is a one-to-one continuous map from Euclidean space to Euclidean space with the same dimension. By invariance of domain this map is a homeomorphism. From this fact we conclude that

$$\tilde{\mathcal{W}}^u(\omega) = \cup_{m \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega, m).$$

□

Propositions 2, 3, 4, 5, 8 and 9 comprise the proof of Theorem 2.3.

By considering the time-reversed flow, we have Theorem 2.4.

6. **Results for overflowing invariant manifolds.** In this section, we recall results for the existence of overflowing and inflowing invariant manifolds and give theorems concerning their foliations. The proofs follow in the same fashion as those for the foliations of stable and unstable manifolds of normally hyperbolic invariant manifolds with slight modifications. We will mainly discuss the conditions of the theorems but not their proofs.

We remark here that random overflowing and inflowing invariant manifolds are similarly defined as for the random invariant manifold and that the normal hyperbolicity for random overflowing and inflowing invariant manifolds could be similarly defined as we did in definition 2.1. For the exact definitions we refer to [15].

We first recall the results on persistence of overflowing manifolds under random perturbations [15]:

Proposition 10. *Assume that $\psi(t)(x)$ is a deterministic C^r flow, $r \geq 1$, which has a compact, connected C^r normally stably hyperbolic overflowing invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$. Then there exists $\rho > 0$ such that for any C^r random flow $\phi(t, \omega, x)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0,1], \omega \in \Omega,$$

and $\bar{\alpha} < r\bar{\beta}$, $\phi(t,\omega)$ has a C^r normally hyperbolic random overflowing invariant manifold $\tilde{\mathcal{M}}(\omega)$ such that for each $\omega \in \Omega$, $\tilde{\mathcal{M}}(\omega)$ is C^r diffeomorphic to $\mathcal{M}$.

Remark 1. If the normal direction contains both stable and unstable directions and is normally hyperbolic, then an unstable manifold exists (see [9] theorem 4) and persists under random perturbation([15]).

For inflowing invariant manifolds we have

Proposition 11. *Assume that $\psi(t)(x)$ is a deterministic C^r flow, $r \geq 1$, which has a compact, connected C^r normally unstably hyperbolic inflowing invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$. Then there exists $\rho > 0$ such that for any C^r random flow $\phi(t, \omega, x)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0,1], \omega \in \Omega,$$

and $\bar{\alpha} < r\bar{\beta}$, $\phi(t,\omega)$ has a C^r normally hyperbolic random inflowing invariant manifold $\tilde{\mathcal{M}}(\omega)$ such that for each $\omega \in \Omega$, $\tilde{\mathcal{M}}(\omega)$ is C^r diffeomorphic to $\mathcal{M}$.

Remark 2. If the normal direction contains both stable and unstable directions and is normally hyperbolic, then a stable manifold exists ([9]) and persists under random perturbation ([15]).

Now, we discuss the foliation results for all four cases above.

Recall that we use the negative invariance property of the random unstable manifold (positive invariance property of the random stable manifold) and the same negative (positive) invariance of the center manifold to unambiguously define a transform on the space of all fibers. In other words, to construct the unique family of fibers, we demand the normal direction and the center direction have the same invariance property.

Under the conditions of Propositions 10, the normal direction is positively invariant while the center direction is negatively invariant, which implies that the graph transform can not be defined for all points and that the neighborhood of the center manifold can not be foliated completely. The case of Proposition 11 is similar.

For the cases in Remarks 1 and 2, we generally do not have the persistence of the center manifold. And we do not have a result about the foliation of the random unstable manifold based on the random center manifold, because there is no random center manifold. In order for the random unstable manifold in Remark 1 (random stable manifold in Remark 2) to be foliated, some extra conditions should be given. We have the following proposition [15]:

Proposition 12. *Assume that $\psi(t)$ is a C^r flow, $r \geq 1$, and has compact, connected C^r normally hyperbolic overflowing (inflowing) invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$ with $\alpha < r\beta$, $\mathcal{M}$ has both stable and unstable manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$. Then there exists $\rho > 0$ such that for any random C^r flow $\phi(t,\omega)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0,1], \omega \in \Omega,$$

then as long as $\phi(t,\omega)$ has compact, connected C^r random overflowing (inflowing) invariant manifold $\bar{\tilde{\mathcal{M}}}(\omega)$ with stable and unstable manifolds $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ such that $\bar{\tilde{\mathcal{M}}}(\omega)$, $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ are C^1 close to $\bar{\mathcal{M}}$, $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. Then $\bar{\tilde{\mathcal{M}}}(\omega)$ is normally hyperbolic with constant $\alpha < r\beta$.

Corresponding to the above proposition, we have the following two theorems:

Theorem 6.1. *Under the conditions of Proposition 12, for the overflowing case, there exists a unique C^{r-1} family of C^r submanifolds $\{\tilde{\mathcal{W}}^{uu}(\omega,x) \ : \ \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ of $\tilde{\mathcal{W}}^u(\omega)$ satisfying:*

(1) For each $(\omega,x) \in \Omega \times \tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{uu}(\omega,x) = \{x\}$, $T_x\tilde{W}^{uu}(\omega,x) = E^u(\omega,x)$ and $\tilde{W}^{uu}(\omega,x)$ varies measurably with respect to (ω,x) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If x_1, $x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \neq x_2$, then $\tilde{\mathcal{W}}^{uu}(\omega,x_1) \cap \tilde{\mathcal{W}}^{uu}(\omega,x_2) = \emptyset$, and

$$\tilde{\mathcal{W}}^u(\omega) = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega,x).$$

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t,\omega)\big(\tilde{\mathcal{W}}^{uu}(\omega,x)\big) \subset \tilde{\mathcal{W}}^{uu}(\theta_t\omega, \phi(t,\omega)x)$ for all $t > 0$ such that $\phi(t,\omega)x \in \tilde{\mathcal{M}}(\theta^t\omega)$.

(4) For $y \in \tilde{\mathcal{W}}^{uu}(\omega,x)$ and $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ with $|\phi(t,\omega)(x_1) - \phi(t,\omega)(x)| \to 0$ as $t \to -\infty$, we have

$$\frac{|\phi(t,\omega)(y) - \phi(t,\omega)(x)|}{|\phi(t,\omega)(y) - \phi(t,\omega)(x_1)|} \to 0$$

exponentially as $t \to -\infty$.

(5) For $y_1, y_2 \in \tilde{\mathcal{W}}^{uu}(\omega,x)$, $|\phi(t,\omega)(y_1) - \phi(t,\omega)(y_2)| \to 0$ exponentially as $t \to -\infty$.

(6) $\tilde{\mathcal{W}}^{uu}(\theta^t\omega,x)$ is C^0 in t for any fixed (ω,x).

Theorem 6.2. *Under the conditions of Proposition 12, for the inflowing case, there exists a unique C^{r-1} family of C^r submanifolds $\{\tilde{\mathcal{W}}^{ss}(\omega,x) \ : \ \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ of $\tilde{\mathcal{W}}^s(\omega)$ satisfying:*

(1) For each $(\omega,x) \in \Omega \times \tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{ss}(\omega,x) = \{x\}$, $T_x\tilde{W}^{ss}(\omega,x) = E^s(\omega,x)$ and $\tilde{W}^{ss}(\omega,x)$ varies measurably with respect to (ω,x) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If x_1, $x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \neq x_2$, then $\tilde{\mathcal{W}}^{ss}(\omega,x_1) \cap \tilde{\mathcal{W}}^{ss}(\omega,x_2) = \emptyset$, and

$$\tilde{\mathcal{W}}^s(\omega) = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{ss}(\omega,x).$$

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t,\omega)\big(\tilde{\mathcal{W}}^{ss}(\omega,x)\big) \subset \tilde{\mathcal{W}}^{ss}(\theta_t\omega, \phi(t,\omega)x)$ for all $t < 0$ such that $\phi(t,\omega)x \in \tilde{\mathcal{M}}(\theta^t\omega)$.

(4) For $y \in \tilde{\mathcal{W}}^{ss}(\omega, x)$ *and* $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ *with* $|\phi(t,\omega)(x_1) - \phi(t,\omega)(x)| \to 0$ *as* $t \to \infty$*, we have*

$$\frac{|\phi(t,\omega)(y) - \phi(t,\omega)(x)|}{|\phi(t,\omega)(y) - \phi(t,\omega)(x_1)|} \to 0$$

exponentially as $t \to +\infty$*.*

(5) For $y_1, y_2 \in \tilde{\mathcal{W}}^{ss}(\omega, x)$*,* $|\phi(t,\omega)(y_1) - \phi(t,\omega)(y_2)| \to 0$ *exponentially as* $t \to +\infty$*.*

(6) $\tilde{\mathcal{W}}^{ss}(\theta^t\omega, x)$ *is* C^0 *in* t *for any fixed* (ω, x)*.*

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