Homework and Pre-Class reading for Math 152H-1 August 6

Homework:

1. Given $\epsilon > 0$. Find N so that for $n > N$ the terms of $a_n = \frac{n^2+2}{4n^2+1}$ satisfy $\left|a_n - \frac{1}{4}\right| < \epsilon$. This is the crucial step in showing that $\lim_{n\to\infty} a_n = \frac{1}{4}$.

2. Show that $b_n = 3n - 1$ diverges to ∞ by using the definition.

3. Use the definition to explain why $a_n \to L$ implies that $C \cdot a_n \to C \cdot L$ where C is a constant (i.e. does not change with n, you're just multiplying all the terms in the sequence by C). Try not to get anxious about the absence of any specific numbers.

4. Prove the squeeze theorem: If $a_n \to L$ and $b_n \to L$ (the same limit) and $a_n \le c_n \le b_n$ then $c_n \to L$. You might want to write $\left|a_n - L\right| < \epsilon$ as $L - \epsilon < a_n < L + \epsilon$, and likewise for b_n and c_n.

5. Use the definition to explain why $1, -1, 1, -1, \ldots$ has no limit. Hint: Suppose the limit is L. Calculate the distance from L to 1 and -1 separately. Show that there is no N_ϵ for $\epsilon = \frac{1}{2}$ so that for *all* $n > N$, etc.

Reading: We now turn from sequences to functions. Recall a function, like $f(x) = x^2$ is some way of assigning a number to each x in a specified set of reals, called the domain of f, and denoted $D(f)$. For instance, $f(x) = x^2$ has domain all of $\mathbb{R}$, but $f(x) = \frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$. A sequence $a_n = \frac{1}{n}$ can be thought of as a function $f(n) = a_n$ whose domain only consists of $n = 1, 2, 3, \ldots$. It is a convenience to think of the domain as being in $\mathbb{R}$, as it allows us to graph the function as usual. However, away from the domain, the function is undefined; it makes no sense to try to evaluate the function at a point not in the domain. Functions can be as nice as $f(x) = 2x + 1$, whose graph is a straight line, or as nasty as

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \text{ is irrational} \\ \frac{1}{q} & \text{if } 0 \le x = \frac{p}{q} \le 1 \text{ is rational} \end{cases}$$

which has domain $[0, 1]$, but is otherwise quite difficult to contemplate. All that is required is that a single value be assigned to each value in the domain.

We now turn to finding limits of functions. There is an additional freedom now. Instead of looking at what happens as $n \to \infty$, we can look at what happens as $x \to a$, where a is any point in the domain of f. To be precise, we assume that all the values of x near a are also in the domain: there is some open interval $(c_1, a) \cup (a, c_2)$ which is in $D(f)$. We do not require that a be in the domain, only that points "near" a be in the domain. Then we ask, do the values of $f(x)$ approach a single, finite value as x gets closer and closer to a. Some examples are in order:

$$\lim_{x\to 2} x^2 + 3x = 2^2 + 3(2) = 10$$

For numbers close to 2, x^2 is close to 2^2, AND as we get closer to 2, x^2 gets closer to 2^2. Note that it is not what happens at 2 that counts, but what happens as we get closer to 2. In these simple examples, it turns out to be the same as plugging in 2, *but that is not always true!!*. Another example:

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$

At 1 both the numerator and the denominator are 0. If we had to use what happened at 1, we'd have no idea what to do. Is it 0 because the numerator is 0, or undefined since the denominator is 0. In fact "$\frac{0}{0}$" is meaningless; when we see that it tells us that we are doing something wrong. So plugging in will not always work. So how do things differently? We use algebra: combine terms that are not yet combined, factor, rationalize denominators, etc. Manipulate the expression until you can get something better. In this case, we can factor using the expression $a^2 - b^2 = (a - b)(a + b)$:

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x\to 1} x + 1 = 2$$

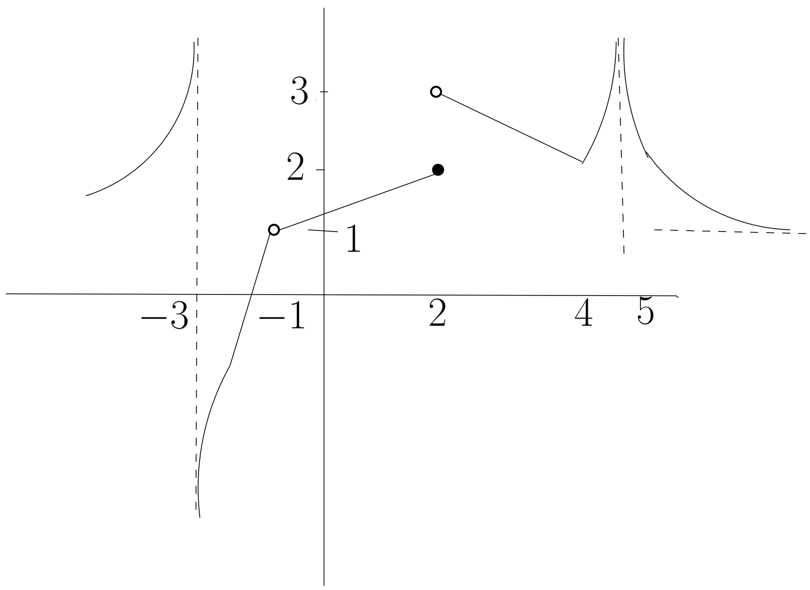
The point is that since we don't care about what happens at 1, the cancellation of the $x - 1$ terms is fine. Away from $x = 1$, these terms are non-zero numbers. Surely though, we have done something by such a drastic change in the function. In fact, we have changed the domain of the function. $\frac{x^2-1}{x-1}$ has domain $(-\infty, 1) \cup (1, \infty)$, since the expression makes no sense when we try to plug in $x = 1$. When we cancel, we find a function $x + 1$ which gives the same values on $(-\infty, 1) \cup (1, \infty)$, but has domain $\mathbb{R}$, and thus tells us what to do at $x = 1$. We'll do more computations shortly. For now we turn to the definitions.

Definitions: As with limits of sequences, we can define the limit of a function in a precise manner. This definition should capture our intuition about limits, while giving us a criterion to check by mathematical manipulations. As with sequences we will need to first guess the limit in order to use the definition. Without further ado, it is:

Definition: $\lim_{x \to a} f(x) = L$ if and only if for each sequence, $\{b_n\}$ converging to a, the sequence $\{f(b_n)\}$ converges to L. Here all of the numbers, b_n, are in some open set $(b, a) \cup (a, c)$ of $D(f)$.

This also provides a simple way to prove that a limit does not exist:

> To show that $\lim_{x \to a} f(x)$ does not exist, find two sequences $\{b_n\}$ and $\{c_n\}$ converging to a, but such that the sequences, $f(a_n)$ and $f(b_n)$, either do not converge, or converge to different limits.



The graph above depicts a function, convince yourself that:

1. $\lim_{x \to -1} f(x) = 1$, i.e that any sequence converging to -1 gives a sequence of function values (points on the graph) converging to 1.
2. $\lim_{x \to 4.5} f(x) = +\infty$
3. $\lim_{x \to -3} f(x)$ and $\lim_{x \to 2} f(x)$ do not exist. Describe two sequences converging to 2 such that $f(a_n)$ and $f(b_n)$ do not converge to the same value.

This definition has the merit of being fairly simple, even if it doesn't seem like it is. But it is somewhat hard to manipulate. Another definition is:

Definition: We will say that $f(x)$ has limit L as x approaches a if for each ϵ, there is a δ such that for each x in $(a - \delta, a) \cup (a, a + \delta)$ we have $\left|f(x) - L\right| < \epsilon$.

Now that we have two definitions, they had better tell us the same thing. How do we see that one implies the other?